Three-dimensional analysis of thermal stresses in smart shells

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The paper focuses on the implementation of the sampling surfaces (SaS) method proposed recently by the authors for the three-dimensional coupled steady-state thermoelectroelastic analysis of layered composite shells with piezoelectric sensors and actuators embedded into the shell body. The SaS shell formulation is based on choosing inside the nth layer I_n not equally spaced SaS parallel to the middle surface in order to introduce the temperatures, electric potentials and displacements of these surfaces as basic shell variables. Such choice of unknowns with the consequent use of Lagrange polynomials of degree $I_n - 1$ in the assumed distributions of the temperature, electric potential, displacements and mechanical properties through the thickness of the layer leads to the robust thermopiezoelectric shell formulation. The SaS are located inside each layer at Chebyshev polynomial nodes that allows one to minimize uniformly the error due to the Lagrange interpolation. As a result, the SaS formulation can be applied efficiently to analytical solutions for layered piezoelectric shells.

Key Words: *Thermoelectroelasticity, Layered piezoelectric shell, 3D thermal stress analysis*

1 Introduction

Three-dimensional (3D) analysis of layered piezoelectric plates and shells under thermal loading has received considerable attention during past twenty years (see, e.g. Ref. [1]). There are at least five approaches to 3D exact solutions of thermoelectroelasticity for piezoelectric plates and shells, namely, the Pagano approach, the state space approach, the power series expansion approach, the asymptotic expansion approach, and the sampling surfaces (SaS) approach.

In this paper, the SaS approach is utilized for the thermoelectroelastic stress analysis of layered piezoelectric shells. According to the SaS approach [2,3], we choose arbitrarily located surfaces inside the nth layer parallel to the middle surface of a shell in order to introduce temperatures $T^{(n)1}, T^{(n)2},..., T^{(n)I_n}$, electric potentials $\varphi^{(n)1}, \varphi^{(n)2},..., \varphi^{(n)I_n}$ and displacement vectors $\mathbf{u}^{(n)1}, \mathbf{u}^{(n)2},..., \mathbf{u}^{(n)I_n}$ of these surfaces as basic shell variables, where I_n is the total number of SaS of the nth layer ($I_n \geq 3$). Such choice of temperatures, electric potentials and displacements with the consequent use of the Lagrange polynomials of degree $I_n - 1$ in the thickness direction for each layer allows the presentation of governing equations of the SaS shell formulation in a very compact form.

It should be noticed that the SaS shell formulation with equally spaced SaS does not work properly with the Lagrange polynomials of high degree because of Runge's phenomenon. This phenomenon yields the wild oscillation at the edges of the interval when the user deals with some specific functions similar to the shell metric functions. If the number of equispaced nodes is increased then the oscillations become even larger. However, the use of the Chebyshev polynomial nodes inside the shell body can help to improve significantly the behavior of the Lagrange polynomials of high degree because such a choice permits one to minimize uniformly the error due to the Lagrange interpolation. This fact gives an opportunity to obtain the displacements and stresses with a prescribed accuracy employing the sufficiently large number of SaS. It means in turn that the solutions based on the SaS formulation *asymptotically* approach the 3D exact solutions of thermoelectroelasticity as $I_n \to \infty$.

2 Description of displacement and strain fields

Consider a layered shell of the thickness h. Let the middle surface Ω be described by orthogonal curvilinear coordinates θ_1 and θ_2, which are referred to the lines of principal curvatures of its surface. The thickness coordinate θ_3 is oriented in the normal direction. Introduce the following notations: $A_\alpha(\theta_1, \theta_2)$ are the coefficients of the first fundamental form; $\kappa_\alpha(\theta_1, \theta_2)$ are the principal curvatures of the middle surface; $c_\alpha = 1 + \kappa_\alpha \theta_3$ are the components of the shifter tensor; $c_\alpha^{(n)i_n}(\theta_1, \theta_2)$ are the components of the shifter tensor at SaS defined as

$$c_\alpha^{(n)i_n} = c_\alpha(\theta_3^{(n)i_n}) = 1 + \kappa_\alpha \theta_3^{(n)i_n}, \tag{1}$$

where $\theta_3^{(n)i_n}$ are the transverse coordinates of SaS of the nth layer $\Omega^{(n)i_n}$ given by

$$\theta_3^{(n)1} = \theta_3^{[n-1]}, \quad \theta_3^{(n)I_n} = \theta_3^{[n]}, \tag{2}$$

$$\theta_3^{(n)m_n} = \frac{1}{2}(\theta_3^{[n-1]} + \theta_3^{[n]}) - \frac{1}{2} h^{(n)} \cos\left(\pi \frac{2m_n - 3}{2(I_n - 2)}\right),$$

where $\theta_3^{[n-1]}$ and $\theta_3^{[n]}$ are the transverse coordinates of layer interfaces $\Omega^{[n-1]}$ and $\Omega^{[n]}$; $h^{(n)} = \theta_3^{[n]} - \theta_3^{[n-1]}$ is the thickness of the nth layer. It is worth noting that the transverse coordinates of inner SaS $\theta_3^{(n)m_n}$ coincide with coordinates of the Chebyshev polynomial nodes. This fact has a great meaning for a convergence of the SaS method [3,4].

Here, the index n identifies the belonging of any quantity to the nth layer and runs from 1 to N, where N is the number of layers; the index m_n identifies the belonging of any quantity to the inner SaS of the nth layer and runs from 2 to $I_n - 1$; the indices i_n, j_n, k_n describe all SaS of the nth layer and run from 1 to I_n; Latin tensorial indices i, j, k, l range from 1 to 3; Greek indices α, β range from 1 to 2.

We start now with the first two assumptions of the proposed layered piezoelectric shell formulation. Let us assume that the displacement and strain fields are distributed through the thickness of the nth layer as

$$u_i^{(n)} = \sum_{i_n} L^{(n)i_n} u_i^{(n)i_n}, \quad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}, \tag{3}$$

$$\varepsilon_{ij}^{(n)} = \sum_{i_n} L^{(n)i_n} \varepsilon_{ij}^{(n)i_n}, \quad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}, \tag{4}$$

where $u_i^{(n)i_n}(\theta_1, \theta_2)$ and $\varepsilon_{ij}^{(n)i_n}(\theta_1, \theta_2)$ are the displacements and strains of SaS of the nth layer $\Omega^{(n)i_n}$; $L^{(n)i_n}(\theta_3)$ are the Lagrange polynomials of degree $I_n - 1$ given by

$$u_i^{(n)i_n} = u_i(\theta_3^{(n)i_n}), \tag{5}$$

$$\varepsilon_{ij}^{(n)i_n} = \varepsilon_{ij}(\theta_3^{(n)i_n}), \tag{6}$$

$$L^{(n)i_n} = \prod_{j_n \ne i_n} \frac{\theta_3 - \theta_3^{(n)j_n}}{\theta_3^{(n)i_n} - \theta_3^{(n)j_n}}. \tag{7}$$

The strains of SaS of the nth layer in terms of displacements of SaS are expressed as

$$2\varepsilon_{\alpha\beta}^{(n)i_n} = \frac{1}{c_\beta^{(n)i_n}} \lambda_{\alpha\beta}^{(n)i_n} + \frac{1}{c_\alpha^{(n)i_n}} \lambda_{\beta\alpha}^{(n)i_n},$$

$$2\varepsilon_{\alpha 3}^{(n)i_n} = \frac{1}{c_\alpha^{(n)i_n}} \lambda_{3\alpha}^{(n)i_n} + \beta_\alpha^{(n)i_n}, \quad \varepsilon_{33}^{(n)i_n} = \beta_3^{(n)i_n}, \tag{8}$$

where $\lambda_{i\alpha}^{(n)i_n}(\theta_1, \theta_2)$ are the strain parameters of SaS of the nth layer introduced in Ref. [3]; $\beta_i^{(n)i_n}(\theta_1, \theta_2)$ are the values of the derivative of displacements with respect to thickness coordinate on SaS defined as

$$\lambda_{\alpha\alpha}^{(n)i_n} = \frac{1}{A_\alpha} u_{\alpha,\alpha}^{(n)i_n} + B_\alpha u_\beta^{(n)i_n} + k_\alpha u_3^{(n)i_n},$$

$$\lambda_{\beta\alpha}^{(n)i_n} = \frac{1}{A_\alpha} u_{\beta,\alpha}^{(n)i_n} - B_\alpha u_\alpha^{(n)i_n},$$

$$\lambda_{3\alpha}^{(n)i_n} = \frac{1}{A_\alpha} u_{3,\alpha}^{(n)i_n} - k_\alpha u_\alpha^{(n)i_n} \text{ for } \beta \ne \alpha,$$

$$\beta_i^{(n)i_n} = u_{i,3}(\theta_3^{(n)i_n}) = \sum_{j_n} M^{(n)j_n}(\theta_3^{(n)i_n}) u_i^{(n)j_n}, \tag{9}$$

where $M^{(n)j_n} = L_{,3}^{(n)j_n}$ are the derivatives of Lagrange polynomials.

It is seen from Eq. (9) that the key functions $\beta_i^{(n)i_n}$ of the layered shell formulation are represented as a linear combination of displacements of SaS of the nth layer $u_i^{(n)j_n}$.

3 Description of electric field

Next, we introduce the third and fourth assumptions of the proposed layered thermo-piezoelectric shell formulation. Let the electric potential and the electric field be distributed through the thickness of the nth layer similar to Eqs (3,4):

$$\varphi^{(n)} = \sum_{i_n} L^{(n)i_n} \varphi^{(n)i_n}, \quad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}, \tag{10}$$

$$E_i^{(n)} = \sum_{i_n} L^{(n)i_n} E_i^{(n)i_n}, \quad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}, \tag{11}$$

where $\varphi^{(n)i_n}(\theta_1, \theta_2)$ are the electric potentials of SaS of the nth layer; $E_i^{(n)i_n}(\theta_1, \theta_2)$ are the components of the electric field at SaS of the nth layer defined as

$$\varphi^{(n)i_n} = \varphi(\theta_3^{(n)i_n}), \tag{12}$$

$$E_i^{(n)i_n} = E_i(\theta_3^{(n)i_n}). \tag{13}$$

The electric field on SaS of the nth layer in terms of electric potentials of SaS is given by

$$E_\alpha^{(n)i_n} = -\frac{1}{A_\alpha c_\alpha^{(n)i_n}} \varphi_{,\alpha}^{(n)i_n}, \tag{14}$$

$$E_3^{(n)i_n} = -\sum_{j_n} M^{(n)j_n}(\theta_3^{(n)i_n}) \varphi^{(n)j_n}. \tag{15}$$

As can be seen from Eq. (15), the normal components of the electric field on SaS of the nth layer $E_3^{(n)i_n}$ are represented as a linear combination of electric potentials of SaS of the same layer $\varphi^{(n)j_n}$.

4 Description of temperature field

The following step consists in a choice of the suitable approximation of the temperature and temperature gradient through the thickness of the layer. It is apparent that the temperature and temperature gradient distributions should be chosen similar to Eqs (3,4) and Eqs (10,11). Therefore, the next two assumptions of the proposed layered thermo-piezoelectric shell formulation are

$$T^{(n)} = \sum_{i_n} L^{(n)i_n} T^{(n)i_n}, \quad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}, \tag{16}$$

$$\Gamma_i^{(n)} = \sum_{i_n} L^{(n)i_n} \Gamma_i^{(n)i_n}, \quad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}, \tag{17}$$

where $T^{(n)i_n}(\theta_1, \theta_2)$ are the temperatures of SaS of the nth layer; $\Gamma_i^{(n)i_n}(\theta_1, \theta_2)$ are the components of the temperature gradient on SaS of the nth layer defined as

$$T^{(n)i_n} = T(\theta_3^{(n)i_n}), \tag{18}$$

$$\Gamma_i^{(n)i_n} = \Gamma_i(\theta_3^{(n)i_n}). \tag{19}$$

The components of the temperature gradient on SaS of the nth layer in terms of temperatures of SaS are expressed as

$$\Gamma_\alpha^{(n)i_n} = \frac{1}{A_\alpha c_\alpha^{(n)i_n}} T_{,\alpha}^{(n)i_n}, \tag{20}$$

$$\Gamma_3^{(n)i_n} = \sum_{j_n} M^{(n)j_n}(\theta_3^{(n)i_n}) T^{(n)j_n}. \tag{21}$$

It is seen from Eq. (21) that the normal components of the temperature gradient at SaS of the nth layer $\Gamma_3^{(n)i_n}$ are represented again as a linear combination of temperatures of SaS of same layer $T^{(n)j_n}$.

5 Constitutive equations

As constitutive equations, we accept the Fourier heat conduction equations

$$q_i^{(n)} = -k_{ij}^{(n)} \Gamma_j^{(n)}, \quad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}, \tag{22}$$

where $k_{ij}^{(n)}$ are the thermal conductivities of the nth layer.

Introduce the seventh assumption of the thermal layered shell formulation. Assume that the thermal conductivity coefficients are distributed through the thickness of the nth layer as follows:

$$k_{ij}^{(n)} = \sum_{i_n} L^{(n)i_n} k_{ij}^{(n)i_n}, \quad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]} \tag{23}$$

that is extensively utilized in this paper, where $k_{ij}^{(n)i_n} = k_{ij}^{(n)}(\theta_3^{(n)i_n})$ are the values of the thermal conductivity tensor on SaS of the nth layer.

For simplicity, we consider the case of linear piezoelectric materials. Therefore, the constitutive equations are expressed as

$$\sigma_{ij}^{(n)} = C_{ijkl}^{(n)} \varepsilon_{kl}^{(n)} - e_{kij}^{(n)} E_k^{(n)} - \gamma_{ij}^{(n)} \Theta^{(n)},$$

$$D_i^{(n)} = e_{ikl}^{(n)} \varepsilon_{kl}^{(n)} + \in_{ik}^{(n)} E_k^{(n)} + r_i^{(n)} \Theta^{(n)},$$

$$\eta^{(n)} = \gamma_{kl}^{(n)} \varepsilon_{kl}^{(n)} + r_k^{(n)} E_k^{(n)} + \chi^{(n)} \Theta^{(n)}, \tag{24}$$

where $C_{ijkl}^{(n)}$ are the elastic constants of the nth layer; $e_{kij}^{(n)}$ are the piezoelectric constants; $\gamma_{ij}^{(n)}$ are the thermal stress coefficients; $\in_{ik}^{(n)}$ are the dielectric constants; $r_i^{(n)}$ are the pyroelectric constants; $\chi^{(n)}$ is the entropy-temperature coefficient defined as

$$\chi^{(n)} = \rho^{(n)} c_{\mathrm{v}}^{(n)} / T_0, \tag{25}$$

where $\rho^{(n)}$ and $c_{\mathrm{v}}^{(n)}$ are the mass density and the specific heat per unit mass of the nth layer at constant strain; T_0 is the reference temperature.

Finally, we introduce the last assumption of the SaS thermopiezoelectric shell formulation. Let the material constants be distributed through the thickness of the nth layer as accepted throughout this paper

$$\Xi^{(n)} = \sum_{i_n} L^{(n)i_n} \Xi^{(n)i_n}, \quad \theta_3^{[n-1]} \le \theta_3 \le \theta_3^{[n]}, \tag{26}$$

$$\Xi^{(n)} = [C_{ijkl}^{(n)}, e_{ijk}^{(n)}, \gamma_{ij}^{(n)}, \in_{ij}^{(n)}, r_i^{(n)}, \rho^{(n)}, c_{\mathrm{v}}^{(n)}],$$

where $\Xi^{(n)i_n} = \Xi^{(n)}(\theta_3^{(n)i_n})$ are the values of material constants on SaS of the nth layer.

6 Analytical solution for layered piezoelectric cylindrical shell

In this section, we study a layered anisotropic cylindrical shell with embedded piezoelectric layers subjected to axisymmetric thermal and electro-mechanical loads. The boundary conditions for the simply supported shell with electrically grounded edges maintained at the reference temperature are written as

$$\Theta^{(n)} = \varphi^{(n)} = \sigma_{11}^{(n)} = \sigma_{12}^{(n)} = u_3^{(n)} = 0 \text{ at } \theta_1 = 0, \theta_1 = L,$$

where $\Theta^{(n)} = T^{(n)} - T_0$ is the temperature rise; θ_1 is the axial coordinate; L is the length of the shell. To satisfy the boundary conditions, we seek the analytical solution by a method of the Fourier series expansion

$$\Theta^{(n)i_n} = \sum_r \Theta_r^{(n)i_n} \sin(\pi\theta_1 / L), \tag{27}$$

$$\varphi^{(n)i_n} = \sum_r \varphi_r^{(n)i_n} \sin(\pi\theta_1 / L), \tag{28}$$

$$u_1^{(n)i_n} = \sum_r u_{1r}^{(n)i_n} \cos(\pi\theta_1 / L), \tag{29}$$

$$u_2^{(n)i_n} = \sum_r u_{2r}^{(n)i_n} \cos(\pi\theta_1 / L), \tag{30}$$

$$u_3^{(n)i_n} = \sum_r \mathrm{u}_{3r}^{(n)i_n} \sin(\pi\theta_1 / L), \tag{31}$$

where r is the wave number. The external electromechanical loads are also expanded in Fourier series.

Substituting Fourier series (27) in a variational equation of the heat conduction theory, we obtain the systems of linear algebraic equations in terms of temperature rises $\Theta_r^{(n)i_n}$ of order K, where $K = \sum_n I_n - N + 1$. Therefore, the temperature rises of SaS of the nth layer can be found by using a method of Gaussian elimination.

Substituting next Fourier series (27)-(31) in a variational equation of thermopiezoelectric shell theory, one obtains the systems of linear algebraic equations in terms of $\Theta_r^{(n)i_n}$, $\varphi_r^{(n)i_n}$, $u_{1r}^{(n)i_n}$, $u_{2r}^{(n)i_n}$ and $u_{3r}^{(n)i_n}$ of order $4K$ because the temperature rises of SaS are already known. These linear systems are solved again through the method of Gaussian elimination.

The described algorithm was performed with the Symbolic Math Toolbox, which incorporates symbolic computations into the numeric environment of MATLAB. This permits the obtaining of analytical solutions for axisymmetric layered anisotropic cylindrical shells in the framework of the SaS thermoelectroelastic shell formulation, which asymptotically approach the 3D exact solutions of thermopiezoelectricity as I_n tends to infinity.

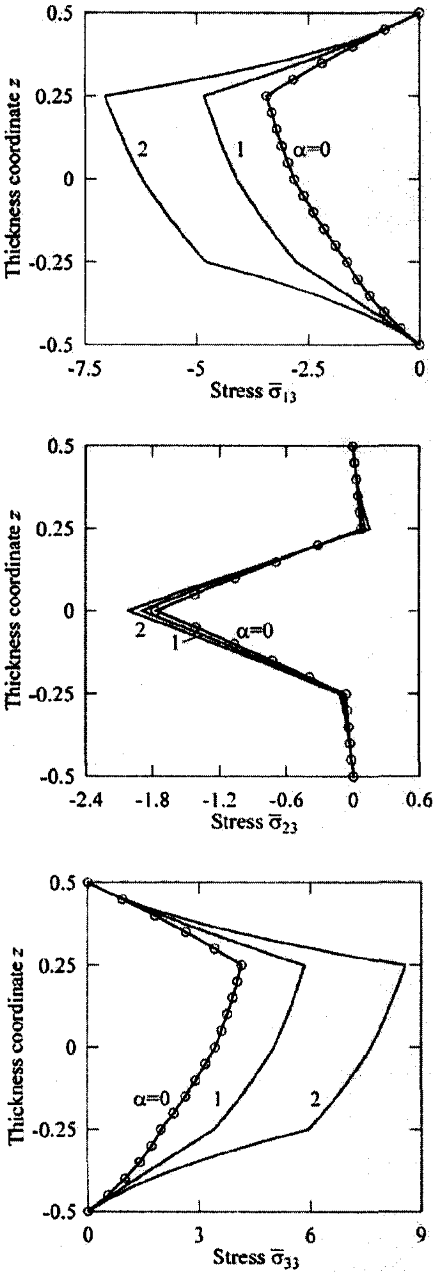

Figure 1. Through-thickness distributions of transverse stresses for a hybrid four-layer anisotropic cylindrical shell with $R/h = 10$ and $I_1 = I_2 = I_3 = I_4 = 9$; exact solution on the basis of the SaS method Ref. [4] (○).

Consider a two-layer angle-ply cylindrical shell [45/–45] composed of the graphite/epoxy composite and covered with functionally graded (FG) piezoelectric layers on its bottom and top surfaces. This means that the hybrid four-layer cylindrical shell [PZT/45/–45/PZT] with ply thicknesses $h_n = h/4$ is considered. The material properties of the graphite-epoxy composite are presented in Ref. [4]. Concerning both FG piezoelectric layers it is assumed that their material properties are distributed through the shell thickness according to the power law:

$$\Phi^{(1)} = \Phi^{(0)}(3+4z)^{\alpha}, \quad -0.5 \le z \le -0.25,$$

$$\Phi^{(4)} = \Phi^{(0)}(3-4z)^{\alpha}, \quad 0.25 \le z \le 0.5,$$

$$\Phi^{(m)} = [C_{ijkl}^{(m)}, e_{ijk}^{(m)}, \gamma_{ij}^{(m)}, \in_{ij}^{(m)}, r_i^{(m)}, \rho^{(m)}, c_v^{(m)}, k_{ij}^{(m)}],$$

where $m = 0,1$ and 4; $C_{ijkl}^{(0)}$, $e_{ijk}^{(0)}$, $\gamma_{ij}^{(0)}$, $\in_{ij}^{(0)}$, $r_i^{(0)}$, $\rho^{(0)}$, $c_v^{(0)}$ and $k_{ij}^{(0)}$ are the material properties of the PZT-5A given in Ref. [4]; α is the material gradient index; $z = \theta_3/h$ is the dimensionless thickness coordinate.

The shell is loaded on the top surface by sinusoidally distributed temperature loading. The boundary conditions on the bottom and top surfaces are

$$\Theta^{+} = \Theta_0 \sin(\pi\theta_1/L), \quad D_3^{+} = \sigma_{13}^{+} = \sigma_{23}^{+} = \sigma_{33}^{+} = 0,$$

$$\Theta^{-} = D_3^{-} = \sigma_{13}^{-} = \sigma_{23}^{-} = \sigma_{33}^{-} = 0,$$

where $\Theta_0 = 1\,\text{K}$ and $T_0 = 293\,\text{K}$. The geometric parameters of a shell are chosen to be $L = 4\,\text{m}$ and $R = 1\,\text{m}$, where R is the radius of the middle surface. Following Ref. [4], we introduce dimensionless stresses $\bar{\sigma}_{i3}$ as functions of the thickness coordinate z.

Fig. 1 shows the distribution of transverse stresses $\bar{\sigma}_{i3}$ through the thickness of the hybrid four-layer cylindrical shell with $R/h = 10$ taking nine SaS for each layer. A comparison with the analytical solution [4] on the basis of the SaS formulation for the two-layer angle-ply cylindrical shell covered with PZT-5A layers (see curves denoted by $\alpha = 0$) is also presented. These results demonstrate convincingly the high potential of the developed thermopiezoelectric shell formulation because the boundary conditions on bottom and top surfaces and the continuity conditions at interfaces for transverse stresses are satisfied properly with a very high accuracy.

References

(1) Wu, C.P., Chiu, K.H. and Wang, Y.M., A review on the three-dimensional analytical approaches of multilayered and functionally graded piezoelectric plates and shells, Comput. Mater. Continua, Vol. 8, 93–132, 2008.

(2) Kulikov, G.M. and Carrera, E., Finite deformation higher-order shell models and rigid-body motions. Int. J. Solids Struct., 45, 3153–3172, 2008.

(3) Kulikov, G.M. and Plotnikova, S.V., Advanced formulation for laminated composite shells: 3D stress analysis and rigid-body motions, Compos. Struct., Vol. 95, 236–246, 2013.

(4) Kulikov, G.M., Mamontov, A.A. and Plotnikova, S.V., Coupled thermoelectroelastic stress analysis of piezoelectric shells, Compos. Struct., Vol. 124, 65–76, 2015.