ISSN 1052-6188, Journal of Machinery Manufacture and Reliability, 2014, Vol. 43, No. 2, pp. 132–139. © Allerton Press, Inc., 2014.
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RELIABILITY, STRENGTH, AND WEAR RESISTANCE OF MACHINES AND STRUCTURES

Analytical Method for Solving Three-Dimensional Thermoelasticity Problems for Composite Shells

G. M. Kulikov and S. V. Plotnikova

Abstract—A new method for solving thermoelasticity problems for thick and thin shells in the three-dimensional formulation is considered. According this method, a shell body is divided into N sampling surfaces parallel to the middle surface and located at Chebyshev polynomial nodes to choose temperatures and vectors of displacements of these surfaces as unknown functions. Such a choice of unknown functions makes it possible to represent governing equations of the proposed theory for composite shells in sufficiently compact form and to use deformation relationships describing correctly the shell displacement as a rigid body.

DOI: 10.3103/S1052618814010129

(i) As is known [1, 2], the traditional way to construct the shell theory is the expansion of displacement into power series with respect to the transverse coordinate θ_3 measured along the outer normal to the middle surface. For the approximate representation of the displacement field, it is possible to use finite segments of power series because the general aim of the elastic shell theory is the derivation of approximate solutions of problems of the three-dimensional theory. However, the apparent advantage of such an approach is lost in the case of its application in problems of statics of thick thermoelastic shells, in which, to obtain admissible results, there is a need to maintain a sufficiently large number of terms in the corresponding expansions.

The more effective approach is associated with the introduction into the shell body of the sampling surfaces $\Omega^1, \Omega^2, \ldots, \Omega^N$ parallel to the middle surface for the purpose of using temperatures $T^1, T^2, \ldots, T^N$ and displacement vectors $\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N$ of these surfaces as known functions [3–5]. Such a choice of known functions with the subsequent use of Lagrange polynomials of degree $N - 1$ in spatial approximations of displacements allows us to represent resolution equations of the proposed theory for high order shells in sufficiently compact form and to construct deformation relationships describing correctly the shell displacement as a rigid body in the curvilinear surface coordinate system [5].

The theory of high order shells [3] is based on the use of equidistant sampling surfaces and shell faces are chosen as the references. This limits the application of this theory for the calculation of thick shells. The point is that the proposed spatial polynomial interpolation of the displacement vector with the use of Lagrange polynomials of high degree may lead, because of Runge's phenomenon, to the significant oscillation of polynomial approximations in the area of the edge effect. This phenomenon was discovered by Runge [6] when the study of the error of polynomial interpolation for approximation of some functions on the uniform grid. As the polynomial degree increases, the interpolation error may go to infinity. The elimination of the phenomenon mentioned in numerical analysis is attained by the use of roots of the Chebyshev polynomial as interpolation nodes [7], which helps to enhance substantially the behavior of polynomial approximations of high degree, for which the interpolation error goes to zero as $N \to \infty$. This gives the possibility of solving three-dimensional problems of statics for thick shells with any prescribed accuracy with a sufficiently large number of sampling surfaces.

(ii) Let us consider a shell with constant thickness h. We assign middle surface Ω to curvilinear orthogonal coordinates θ_1 and θ_2 measured along the lines of principal curvature and measure coordinate θ_3 in the transverse direction. Let $\mathbf{e}_\alpha$ be unit vectors of tangents to coordinate lines θ_α; $\mathbf{e}_3$ be the unit vector of the outer normal to the middle surface; A_α be the coefficients of the first quadratic form; k_α the principal curvatures; $c_\alpha = 1 + k_\alpha\theta_3$ the components of the geometrical shift tensor; $c_\alpha^I = 1 + k_\alpha\theta_3^I$ the components of the geometrical shift tensor at sampling surfaces Ω^I (Fig. 1); and θ_3^I the transverse coordinate of sur-

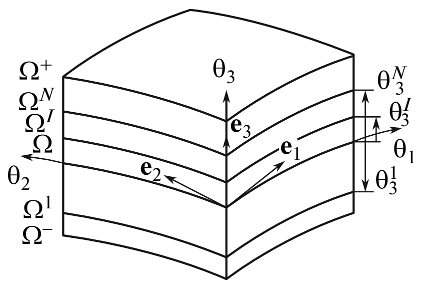

Fig. 1.

faces Ω^I, which are placed within the range $(-h/2, h/2)$ at Chebyshev polynomial nodes with degree N and are determined according to [8] by formula

$$\theta_3^I = -\frac{h}{2}\cos\left(\pi\frac{2I-1}{2N}\right),$$

where I, J, and K denote any values belonging to the sampling surface and take values of 1, 2, ..., N; Greek subscripts $\alpha, \beta = 1, 2$; and Roman subscripts $i, j, k, m = 1, 2, 3$. Note that the present paper uses the convention of summation by repetitive Roman subscripts.

Temperature gradient Γ_i and temperature T are associated by formulas

$$\Gamma_\alpha = \frac{1}{A_\alpha c_\alpha}T_{,\alpha}, \quad \Gamma_3 = T_{,3}. \tag{1}$$

The first assumption of this thermoelastic shell theory, which is based on the sampling surface method, refers to the form of the distribution of temperature and temperature gradient in the transverse direction

$$T = \sum_I L^I T^I, \quad T^I = T(\theta^I), \tag{2}$$

$$\Gamma_i = \sum_I L^I \Gamma_i^I, \quad \Gamma_i^I = \Gamma_i(\theta^I), \tag{3}$$

where T^I, Γ_i^I are the temperature and the temperature gradient at sampling surfaces Ω^I; $L^I(\theta_3)$ are Lagrange polynomials with degree $N-1$: $L^I = \prod_{J \neq I}\frac{\theta_3 - \theta_3^J}{\theta_3^I - \theta_3^J}$.

From relationships (1), (2) is

$$\Gamma_\alpha^I = \frac{1}{A_\alpha c_\alpha^I}T_{,\alpha}^I, \quad \Gamma_3^I = \sum_J M^J(\theta_3^I)T^J, \tag{4}$$

where $M^I = L_{,3}^I$ are polynomials of degree $N-2$; their values at sampling surfaces Ω^I are determined by formulas

$$M^J(\theta_3^I) = \frac{1}{\theta_3^J - \theta_3^I}\prod_{K \neq I, J}\frac{\theta_3^I - \theta_3^K}{\theta_3^J - \theta_3^K} \quad (J \neq I), \quad M^I(\theta_3^I) = -\sum_{J \neq I} M^J(\theta_3^I).$$

Thus, transverse components of temperature gradient Γ_3^I at surfaces Ω^I are represented according to (4) in the form of the linear combination of temperatures T^J of sampling surfaces.

(iii) Components of the deformation tensor at sampling surfaces can be written in the vector form [3, 4]

$$2\varepsilon_{\alpha\beta}^I = \frac{1}{A_\alpha c_\alpha^I}\mathbf{u}_{,\alpha}^I \cdot \mathbf{e}_\beta + \frac{1}{A_\beta c_\beta^I}\mathbf{u}_{,\beta}^I \cdot \mathbf{e}_\alpha, \quad 2\varepsilon_{\alpha 3}^I = \boldsymbol{\beta}^I \cdot \mathbf{e}_\alpha + \frac{1}{A_\alpha c_\alpha^I}\mathbf{u}_{,\alpha}^I \cdot \mathbf{e}_3, \quad \varepsilon_{33}^I = \boldsymbol{\beta}^I \cdot \mathbf{e}_3, \tag{5}$$

where $\mathbf{u}^I = \mathbf{u}(\theta_3^I)$ and $\boldsymbol{\beta}^I = \mathbf{u}_{,3}(\theta_3^I)$ are displacement vectors of sampling surface and the derivative of the displacement vector with respect to coordinate θ_3 at surfaces Ω^I.

Let us represent vectors $\mathbf{u}^I$ and $\boldsymbol{\beta}^I$ in an orthonormal basis $\mathbf{e}_i$ as

$$\mathbf{u}^I = u_i^I \mathbf{e}_i, \quad \boldsymbol{\beta}^I = \beta_i^I \mathbf{e}_i. \tag{6}$$

Differentiating (6) with respect to coordinate θ_3 and taking into account results [2], we obtain

$$\mathbf{u}^I_{,\alpha} = A_\alpha \lambda^I_{i\alpha} \mathbf{e}_i, \tag{7}$$

$$\lambda^I_{\alpha\alpha} = \frac{1}{A_\alpha} u^I_{\alpha,\alpha} + B_\alpha u^I_\beta + k_\alpha u^I_3, \quad \lambda^I_{\beta\alpha} = \frac{1}{A_\alpha} u^I_{\beta,\alpha} - B_\alpha u^I_\alpha \quad (\beta \neq \alpha)$$

$$\lambda^I_{3\alpha} = \frac{1}{A_\alpha} u^I_{3,\alpha} - k_\alpha u^I_\alpha, \quad B_\alpha = \frac{1}{A_\alpha A_\beta} A_{\alpha,\beta} \quad (\beta \neq \alpha). \tag{8}$$

Introducing expansions (6), (7) into (5), we come to the scalar form of the deformation relationships

$$2\varepsilon^I_{\alpha\beta} = \frac{1}{c^I_\beta} \lambda^I_{\alpha\beta} + \frac{1}{c^I_\alpha} \lambda^I_{\beta\alpha}, \quad 2\varepsilon^I_{\alpha 3} = \beta^I_\alpha + \frac{1}{c^I_\alpha} \lambda^I_{3\alpha}, \quad \varepsilon^I_{33} = \beta^I_3. \tag{9}$$

The following step consists in the choice of the distribution law for displacements and deformations over the shell thickness. It is evident that their distribution in the transverse direction should be coordinated with the distribution of the temperature and the temperature gradient (2), (3), i.e.,

$$u_i = \sum_I L^I u^I_i, \tag{10}$$

$$\varepsilon_{ij} = \sum_I L^I \varepsilon^I_{ij}. \tag{11}$$

From relationships (6), (10) we find

$$\beta^I_i = \sum_J M^J(\theta^I_3) u^J_i. \tag{12}$$

It is evident that determining functions β^I_i of this shell theory are represented in the form of the linear combination of displacements u^J_i of the sampling surfaces.

Note that deformation relationships (9), (11) are correctly represented the shell displacements as the rigid body in the system of the curvilinear surface coordinates.

(iv) The variational heat conduction equation has the form:

$$\delta J = 0, \tag{13}$$

$$J = \frac{1}{2} \iint_\Omega \int_{-h/2}^{h/2} q_i \Gamma_i A_1 A_2 c_1 c_2 d\theta_1 d\theta_2 d\theta_3 - \iint_{\bar{\Omega}} T Q_n d\Omega, \tag{14}$$

where q_i are components of the heat flux; Q_n is the heat flux in the direction normal to surface $\bar{\Omega} = \Omega^- + \Omega^+ + \Sigma$ (Ω^-, Ω^+ are shell faces); and Σ is the lateral boundary surface.

Substituting the temperature gradient distribution (3) into (14) and introducing resultants of the heat flux

$$R^I_i = \int_{-h/2}^{h/2} q_i L^I c_1 c_2 d\theta_3, \tag{15}$$

we come to formula

$$J = \frac{1}{2} \iint_\Omega \sum_I R^I_i \Gamma^I_i A_1 A_2 d\theta_1 d\theta_2 - \iint_{\bar{\Omega}} T Q_n d\Omega. \tag{16}$$

The Fourier heat conduction state equation is represented in the form:

$$q_i = -k_{ij}\Gamma_j, \tag{17}$$

where k_{ij} are thermal conductivity coefficients.

Introducing equation (17) of state into (15) and accounting for distribution (3), we obtain

$$R_i^I = -\sum_J \Lambda^{IJ} k_{ij} \Gamma_j^J, \tag{18}$$

$$\Lambda^{IJ} = \int_{-h/2}^{h/2} L^I L^J c_1 c_2 d\theta_3. \tag{19}$$

(v) The variational thermoelasticity equation in the case of conservative loading has the form:

$$\delta\Pi = 0, \tag{20}$$

$$\Pi = \frac{1}{2}\iint_\Omega \int_{-h/2}^{h/2} (\sigma_{ij}\varepsilon_{ij} - S\Theta) A_1 A_2 c_1 c_2 d\theta_1 d\theta_2 d\theta_3 - W, \tag{21}$$

$$W = \iint_\Omega (c_1^+ c_2^+ p_i^+ u_i^+ - c_1^- c_2^- p_i^- u_i^-) A_1 A_2 d\theta_1 d\theta_2 + W_\Sigma, \tag{22}$$

where σ_{ij} is the stress tensor; S is the entropy density; p_i^- , p_i^+ are surface loads acting at shell faces Ω^-, Ω^+; $u_i^- = u_i(-h/2)$, $u_i^+ = u_i(h/2)$ are displacements of shell faces; $c_\alpha^- = 1 - k_\alpha h/2$, $c_\alpha^+ = 1 + k_\alpha h/2$ are components of the displacement tensor at faces; W_Σ is the work of external loads acting at side surface Σ; and Θ is the temperature increment determined by formula

$$\Theta = T - T_0. \tag{23}$$

Substituting the deformation distribution in the transverse direction (11) and temperature distribution

$$\Theta = \sum_I L^I \Theta^I \tag{24}$$

immediately following from (2), (23) into functional (21) and introducing the resultants of stresses and entropy

$$H_{ij}^I = \int_{-h/2}^{h/2} \sigma_{ij} L^I c_1 c_2 d\theta_3, \tag{25}$$

$$P^I = \int_{-h/2}^{h/2} S L^I c_1 c_2 d\theta_3, \tag{26}$$

we obtain

$$\Pi = \frac{1}{2}\iint_\Omega \sum_I (H_{ij}^I \varepsilon_{ij}^I - P^I \Theta^I) A_1 A_2 d\theta_1 d\theta_2 - W. \tag{27}$$

Thermoelasticity state equations [8] are represented in the form:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \gamma_{ij}\Theta, \tag{28}$$

$$S = \gamma_{ij}\varepsilon_{ij} + \rho c_V \Theta / T_0, \tag{29}$$

Table

Sampling surfaces are placed at Chebyshev polynomial nodes							
N	$\bar{\theta}(0.5)$	$\bar{u}_3(0)$	$\bar{\sigma}_{11}(0.5)$	$\bar{\sigma}_{22}(0.5)$	$\bar{\sigma}_{13}(0.25)$	$\bar{\sigma}_{23}(-0.25)$	$\bar{\sigma}_{33}(-0.25)$
3	–1.314	–5.275	9.066	1.604	1.590	2.897	–2.747
5	–1.405	–5.668	8.874	1.856	1.630	3.356	0.106
7	–1.405	–5.660	8.769	1.736	1.667	3.362	1.632
9	–1.405	–5.661	8.765	1.731	1.665	3.361	1.570
11	–1.405	–5.661	8.765	1.730	1.665	3.360	1.570
13	–1.405	–5.661	8.765	1.730	1.665	3.360	1.571
15	–1.405	–5.661	8.765	1.730	1.665	3.360	1.571
Sampling surfaces are placed at equidistant nodes [3]							
N	$\bar{\theta}(0.5)$	$\bar{u}_3(0)$	$\bar{\sigma}_{11}(0.5)$	$\bar{\sigma}_{22}(0.5)$	$\bar{\sigma}_{13}(0.25)$	$\bar{\sigma}_{23}(-0.25)$	$\bar{\sigma}_{33}(-0.25)$
3	–1.337	–5.438	9.240	1.720	1.622	2.968	–2.838
5	–1.406	–5.673	8.894	1.879	1.625	3.313	–0.041
7	–1.405	–5.662	8.777	1.745	1.669	3.368	1.740
9	–1.405	–5.661	8.772	1.738	1.667	3.379	1.561
11	–1.405	–5.662	8.772	1.739	1.663	3.334	1.490
13	–1.405	–5.663	8.774	1.740	1.667	3.366	1.670
15	–1.406	–5.665	8.778	1.742	1.669	3.395	1.563

where C_{ijkm} is the elasticity modulus tensor; γ_{ij} are temperature stresses; and ρ, c_V are the specific density and the specific volumetric heat capacity [9].

Let us introduce stress (28) and entropy (29) into (25), (26), respectively, and accounting for the distribution of deformations and temperature in the transverse direction (11), (24), we come to formulas to calculate the resultants of stresses and entropy

$$H_{ij}^I = \sum_J \Lambda^{IJ}(C_{ijkl}\varepsilon_{kl}^J - \gamma_{ij}\Theta^J), \tag{30}$$

$$P^I = \sum_J \Lambda^{IJ}(\gamma_{ij}\varepsilon_{ij}^J + \rho c_V \Theta^J / T_0). \tag{31}$$

(vi) As an example, consider the bending of a hinged orthotropic shell with size $L/R = 4$ under the action of a heat flux distributed at the outer surface by the sinusoidal law, whereas the inner surface is heat-insulated

$$q_3^+ = q_0 \sin\frac{\pi\theta_1}{L}\cos 2\theta_2, \quad q_3^- = 0,$$

where L is the shell length; R is the radius of the middle surface; and θ_1, θ_2 are the meridional and circumferential coordinates.

The boundary conditions at shell ends are

$$\sigma_{11} = u_2 = u_3 = \Theta = 0 \quad \text{at} \quad \theta_1 = 0, \quad \theta_1 = L. \tag{32}$$

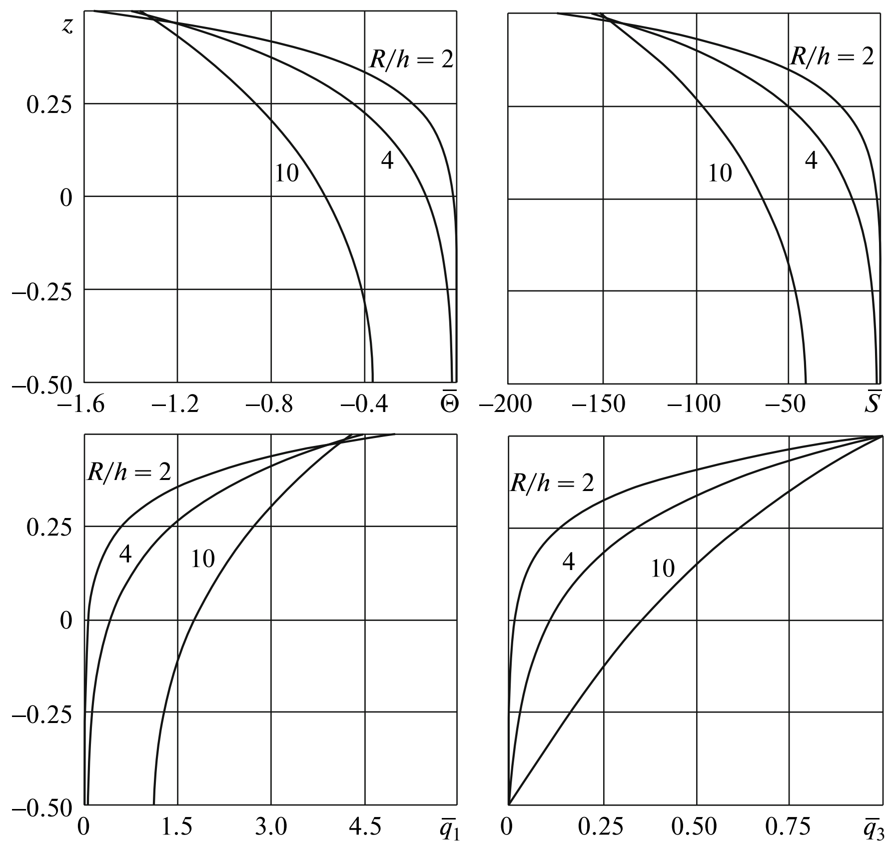

Fig. 2. Distribution of temperature, entropy, and heat flux through the thickness of the composite cylindrical shell.

To satisfy boundary conditions (32) we take

$$\begin{aligned} u_1^I &= u_{10}^I \cos\frac{\pi\theta_1}{L}\cos 2\theta_2, \quad u_2^I = u_{20}^I \sin\frac{\pi\theta_1}{L}\sin 2\theta_2, \\ u_3^I &= u_{30}^I \sin\frac{\pi\theta_1}{L}\cos 2\theta_2, \quad \Theta^I = \Theta_0^I \sin\frac{\pi\theta_1}{L}\cos 2\theta_2. \end{aligned} \tag{33}$$

Substituting (33) into functionals (16), (22), and (27) and taking into account relationships (4), (8), (9), (12), (18), (19), (23), (30), (31) and $W_\Sigma = 0$, we obtain

$$J = J(\Theta_0^I), \quad \Pi = \Pi(\Theta_0^I, u_{i0}^I). \tag{34}$$

From variational equations (13), (20) with allowance made for (34), sets of linear algebraic equations of orders N and $3N$ can be derived

$$\frac{\partial J}{\partial \Theta_0^I} = 0, \quad \frac{\partial \Pi}{\partial u_{i0}^I} = 0. \tag{35}$$

Sets (35) are solved independently of each other by the Gauss method. The described algorithm is realized in the MATLAB programming environment with the use of the ToolBox Symbolic Math software package allowing symbolic computations to be carried out. As a result, we obtain the exact solution for the three-dimensional thermoelasticity problem for the hinged cylindrical shell.

Let the cylindrical shell be made of circumferentially-reinforced composite with the following mechanical parameters: $E_L = 10E_0$, $E_T = E_0$, $G_{LT} = 0.5E_0$, $G_{TT} = 0.2E_0$, $\nu_{LT} = \nu_{TT} = 0.25$, $\alpha_L = \alpha_0$, $\alpha_T = 7.2\alpha_0$, $k_L = 100k_0$, $k_T = k_0$, $\rho = 1800$ kg/m^3, and $c_V = 900$ J/(kg K), where $E_0 = 20$ GPa, $\alpha_0 = 5 \times 10^{-6}$ 1/K, and $k_0 = 0.5$ W/(m K). Usually subscripts L and T correspond to the reinforcement direction and the

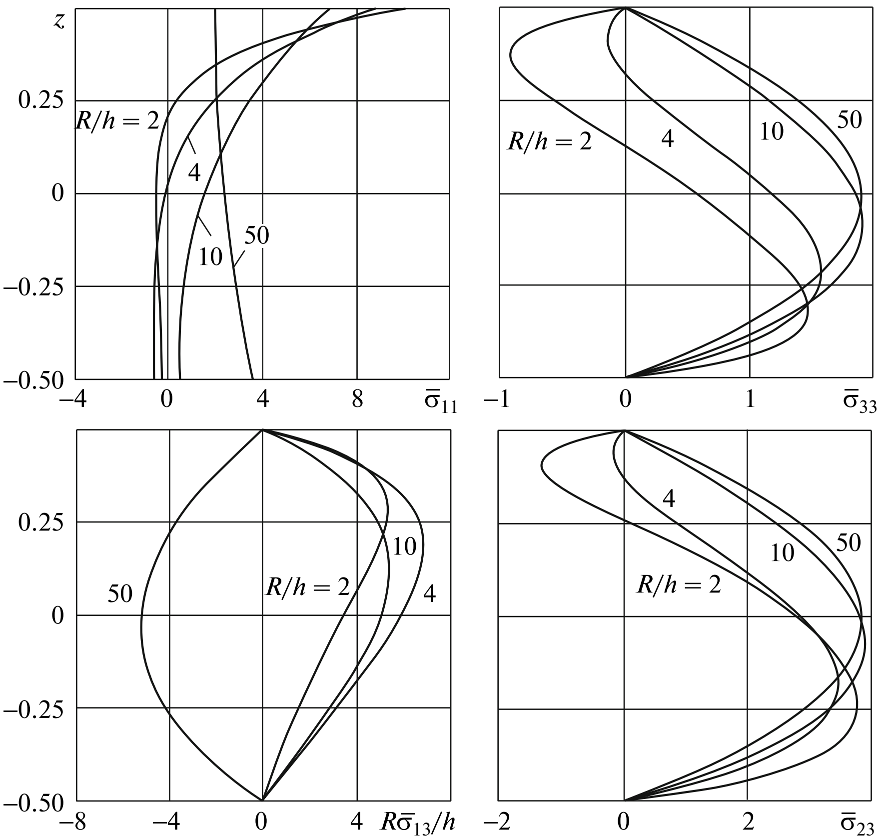

Fig. 3. Distribution of stresses through the thickness of the composite cylindrical shell.

transverse direction. Further, we take $R = 1$ m, $T_0 = 293$ K, $q_0 = 1$ W/m^2 and introduce dimensionless values

$$\begin{gathered}\bar{u}_3 = 10^3 k_0 u_3(L/2, 0, z)/L^2\alpha_0 q_0, \quad \bar{\sigma}_{11} = 10^2 k_0 \sigma_{11}(L/2, 0, z)/LE_0\alpha_0 q_0,\\ \bar{\sigma}_{22} = 10^2 k_0 \sigma_{22}(L/2, 0, z)/LE_0\alpha_0 q_0, \quad \bar{\sigma}_{13} = 10^3 k_0 \sigma_{13}(0, 0, z)/LE_0\alpha_0 q_0,\\ \bar{\sigma}_{23} = 10^3 k_0 \sigma_{23}(L/2, \pi/4, z)/LE_0\alpha_0 q_0, \quad \bar{\sigma}_{33} = 10^3 k_0 \sigma_{33}(L/2, 0, z)/LE_0\alpha_0 q_0,\\ \bar{\Theta} = 10^2 k_0 \Theta(L/2, 0, z)/Lq_0, \quad \bar{S} = k_0 S(L/2, 0, z)/LE_0\alpha_0^2 q_0,\\ \bar{q}_1 = 10^2 q_1(0, 0, z)/q_0, \quad \bar{q}_3 = q_3(L/2, 0, z)/q_0, \quad z = \theta_3/h.\end{gathered} \tag{36}$$

Table data obtained by the choice of sampling surfaces at Chebyshev polynomial nodes, and with the use of equidistant sampling surfaces [3], show that, with sufficiently large parameters N, we can achieve good agreement with both approaches, even for shell thickness ($R/h = 4$). However, uniform convergence to the exact problem solution is absent in approach [3]. Figures 2 and 3 represent the distribution of dimensionless values (36) over the shell thickness with geometrical parameters $R/h = 2, 4, 10, 50$ in the case of the choice of 13 sampling surfaces placed at Chebyshev polynomial nodes, which testify to the high potential of the proposed method for solving the thermoelasticity problem for thick and thin shells in the three-dimensional formulation. It is evident that the boundary conditions at shell faces for transverse components of the stress tensor and the heat flux vector are met with the high accuracy. In addition, Fig. 4 shows logarithmic errors $\delta_i = \log|\bar{\sigma}_{i3}|$ of the fulfillment of boundary conditions for these stresses at shell faces at various parameters N. Figure 4a shows results for the problem solving based on the choice of sampling surfaces at Chebyshev polynomial nodes; and Fig. 4b results with the use of approach [3]. Computations based on approach [3] do not provide the monotonic convergence of the solution and lead to an

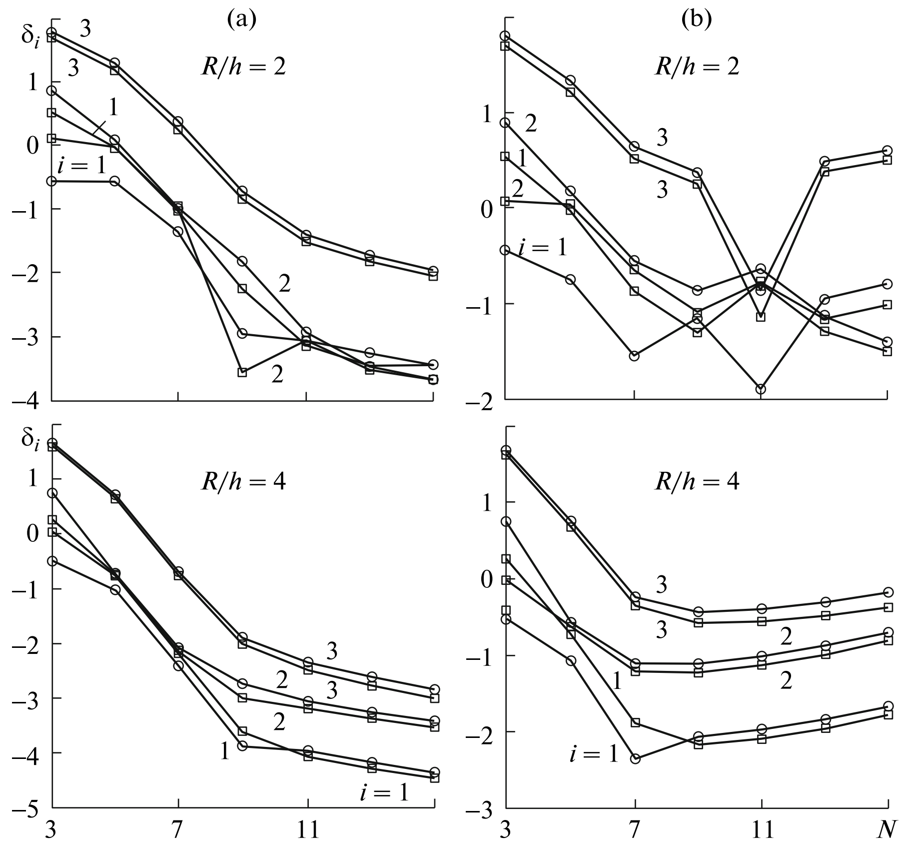

Fig. 4. Logarithmic errors for the computation of transverse stresses δ_i ($i = 1, 2, 3$) at the inner surface (curves with circles) and the external surface (curves with squares) of the cylindrical shell with geometrical parameters $R/h = 2, 4$ based on the present method (a) and the method for equidistant sampling surfaces (b).

inadequate description of the stress state of the shell in the area of the edge effect for a Lagrange polynomial of high degree.

ACKNOWLEDGMENTS

The work was supported by the Russian Foundation for Basic Research (project no. 13-01-00155).

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Translated by S. Ordzhonikidze