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STRAIN-DISPLACEMENT RELATIONSHIPS THAT EXACTLY REPRESENT LARGE RIGID DISPLACEMENTS OF A SHELL

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The finite element method (FEM) has become the most powerful numerical method that is utilized currently for solving problems in mechanics of shells. At the same time, the problem of the construction of curved finite elements for thin shells subject to large displacements and arbitrarily large rotations is still far from being solved [1-3]. It is for this reason that the representation provided by the strain-displacement relationships for rigid displacements of the shell element is inadequate. Therefore, it is no wonder that since the strain relationships that would be able to represent an arbitrary strain state in local curvilinear coordinates are not available in the literature, the degenerate (isoparametric) element concept [4] has been mostly developed. This concept enables one to represent rigid displacements of the element in the global Cartesian reference frame but requires considerable increase in the computational time [3, 5].

In the literature, there are a lot of versions of both the classical (Kirchhoff–Love) theory of shells and the theory refined on the basis of Timoshenko's hypothesis. However, only some of these theories (e.g., those of [6–9]) provide adequate approximations for rigid displacements of the shell. This observation is validated in [7, 10, 11] for the geometrically linear Kirchhoff–Love theory of shells and in [12] for the refined theory of shells. In the present paper, the results of [12] are extended to shells subject to large displacements and arbitrarily large rotations.

1. STRAIN-DISPLACEMENT RELATIONSHIPS OF GEOMETRICALLY LINEAR ELASTICITY

Consider a shell of constant thickness h. As the reference surface S we take an internal surface of the shell. We introduce on the reference surface the curvilinear coordinates α_1 and α_2 measured along the principal curvatures. The transverse coordinate α_3 is measured along the outward normal to the surface S (Fig. 1). Let $\mathbf{e}_1$ and $\mathbf{e}_2$ denote the unit vectors tangent to the coordinate lines α_1 and α_2, $\mathbf{e}_3$ the unit vector of the outward normal, A_α and k_α the Lamé parameters and the curvatures of the coordinate lines of the surface S, δ^- and δ^+ the distances from the surface S to the lower, S^-, and upper, S^+, surfaces of the shell, respectively, and ε_{ij} are the components of the Green–Lagrange strain tensor. Here and in what follows, Latin indices i and j take on the values 1, 2 or 3, and Greek indices α, β, and γ the values 1 or 2.

We will represent the strain-displacement relationships of the geometrically nonlinear theory of elasticity [13] in the vector form

$$2\varepsilon_{ij} = \frac{1}{H_i}\mathbf{u}_{,i}\left(\mathbf{e}_j + \frac{1}{2H_j}\mathbf{u}_{,j}\right) + \frac{1}{H_j}\mathbf{u}_{,j}\left(\mathbf{e}_i + \frac{1}{2H_i}\mathbf{u}_{,i}\right), \qquad \mathbf{u} = \sum_i u_i\mathbf{e}_i, \quad H_\alpha = A_\alpha(1 + k_\alpha\alpha_3), \quad H_3 = 1, \tag{1.1}$$

where $u_i(\alpha_1, \alpha_2, \alpha_3)$ are the displacement vector components, and H_α are the Lamé parameters of a surface all point of which are equidistant from the reference surface S. (Such surfaces sometimes are referred to as parallel surfaces.) The superscript i after the comma stands for partial differentiation with respect to the coordinate α_i.

Arbitrarily large rigid displacements of the shell will be also represented in the vector form

$$\mathbf{u}^R = \boldsymbol{\Delta} + (\mathbf{B} - \mathbf{E})\mathbf{R}, \qquad \mathbf{R} = \mathbf{r} + \alpha_3\mathbf{e}_3, \quad \boldsymbol{\Delta} = \sum_i \Delta_i\mathbf{e}_i, \tag{1.2}$$

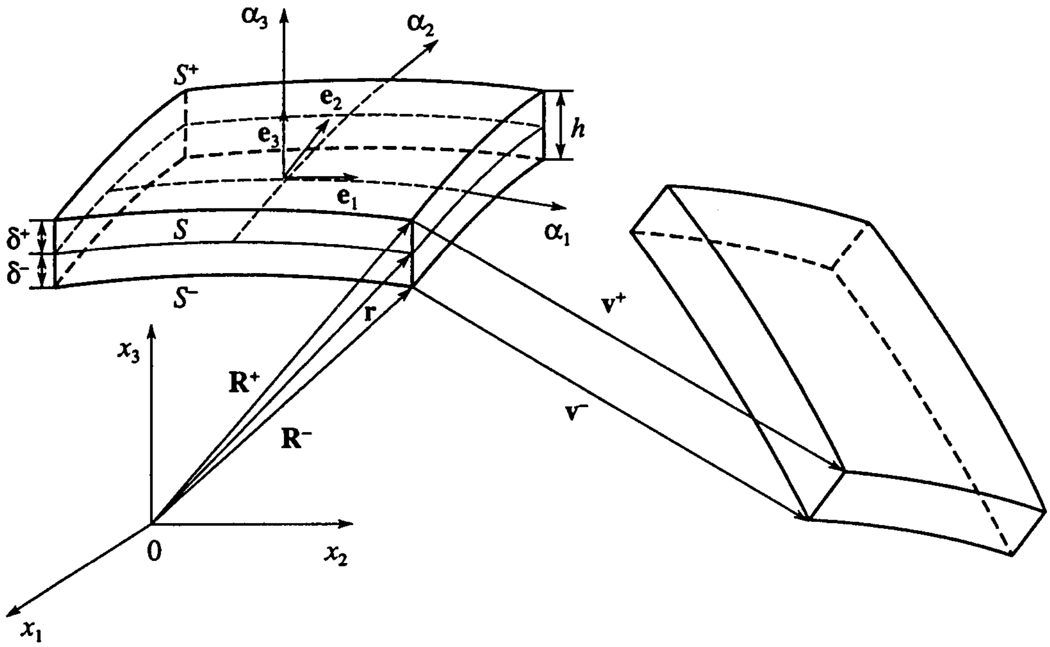

Fig. 1

$$\mathbf{B} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\varphi\sin\psi + \sin\varphi\sin\theta\cos\psi & \cos\varphi\cos\psi + \sin\varphi\sin\theta\sin\psi & \sin\varphi\cos\theta \\ \sin\varphi\sin\psi + \cos\varphi\sin\theta\cos\psi & -\sin\varphi\cos\psi + \cos\varphi\sin\theta\sin\psi & \cos\varphi\cos\theta \end{bmatrix}, \tag{1.3}$$

where $\mathbf{R}$ is the position vector of a point of the shell, $\mathbf{r}(\alpha_1, \alpha_2)$ is the position vector of the orthogonal projection of this point onto the reference surface S, $\mathbf{\Delta}$ is the translational displacement of the shell, $\mathbf{E}$ is the unit matrix, $\mathbf{B}$ is the orthogonal rotation matrix, and φ, θ, and ψ are the Euler–Krylov angles that characterize the rigid rotation of the shell about the point O (Fig. 1).

The differentiation of (1.2) with respect to α_i with reference to (1.3) and the relations $\mathbf{\Delta}_{,i} = \mathbf{0}$ and $\mathbf{R}_{,i} = H_i\mathbf{e}_i$ [7] yields

$$\mathbf{u}^R_{,i} = H_i(\mathbf{B}\mathbf{e}_i - \mathbf{e}_i). \tag{1.4}$$

By substituting the derivatives of (1.4) into the strain-displacement relationships of (1.1) we obtain (after simple transformations)

$$2\varepsilon^R_{ij} = (\mathbf{B}\mathbf{e}_i)(\mathbf{B}\mathbf{e}_j) - \mathbf{e}_i\mathbf{e}_j. \tag{1.5}$$

With reference to the fact that an orthogonal transformation preserves the scalar product, the relation of (1.5) implies

$$\varepsilon^R_{ij} = 0. \tag{1.6}$$

Hence, as could be anticipated, all components of the Green–Lagrange strain tensor in the local curvilinear coordinates α_i are equal to zero for arbitrarily large rigid displacements of the shell.

2. STRAIN-DISPLACEMENT RELATIONSHIPS FOR THE GEOMETRICALLY NONLINEAR REFINED THEORY OF SHELLS

We will make use of Timoshenko's modified kinematic hypothesis according to which the displacements are linearly distributed across the shell thickness [9], specifically,

$$\mathbf{u} = N^-(\alpha_3)\mathbf{v}^- + N^+(\alpha_3)\mathbf{v}^+, \qquad \mathbf{v}^\pm = \sum_i v_i^\pm \mathbf{e}_i, \quad N^-(\alpha_3) = \frac{\delta^+ - \alpha_3}{h}, \quad N^+(\alpha_3) = \frac{\alpha_3 - \delta^-}{h}, \tag{2.1}$$

where $\mathbf{v}^\pm$ are the displacement vectors of the face surfaces of the shell, $S^\pm$, $v_i^\pm(\alpha_1, \alpha_2)$ are the components of these vectors, and $N^\pm(\alpha_3)$ are linear functions that specify the shape of the shell.

Substitute the expressions of (2.1) for displacements into the strain-displacement relationships of (1.1) of the spatial theory of elasticity. With reference to the formulas of [7] for differentiation of the basis vectors, $\mathbf{e}_{3,\alpha} = A_\alpha k_\alpha \mathbf{e}_\alpha$, we arrive at the strain-displacement relationships of the refined geometrically nonlinear theory of medium-thickness shells,

$$2\varepsilon^a_{\alpha\beta} = \left[N^-(\alpha_3)\frac{1}{H_\alpha}\mathbf{v}^-_{,\alpha} + N^+(\alpha_3)\frac{1}{H_\alpha}\mathbf{v}^+_{,\alpha}\right]\mathbf{e}_\beta + \left[N^-(\alpha_3)\frac{1}{H_\beta}\mathbf{v}^-_{,\beta} + N^+(\alpha_3)\frac{1}{H_\beta}\mathbf{v}^+_{,\beta}\right]\mathbf{e}_\alpha$$
$$+ \left[N^-(\alpha_3)\frac{1}{H_\alpha}\mathbf{v}^-_{,\alpha} + N^+(\alpha_3)\frac{1}{H_\alpha}\mathbf{v}^+_{,\alpha}\right]\left[N^-(\alpha_3)\frac{1}{H_\beta}\mathbf{v}^-_{,\beta} + N^+(\alpha_3)\frac{1}{H_\beta}\mathbf{v}^+_{,\beta}\right], \quad (2.2)$$

$$2\varepsilon^a_{\alpha 3} = \frac{\overline{H}_\alpha}{H_\alpha}\boldsymbol{\beta}\mathbf{e}_\alpha + \frac{1}{H_\alpha}\overline{\mathbf{v}}_{,\alpha}(\mathbf{e}_3 + \boldsymbol{\beta}) + (\alpha_3 - \overline{\delta})\frac{1}{H_\alpha}\varepsilon^a_{33,\alpha}, \qquad \overline{H}_\alpha = A_\alpha(1 + k_\alpha\overline{\delta}), \quad \overline{\delta} = \frac{\delta^- + \delta^+}{2}, \quad (2.3)$$

$$\varepsilon^a_{33} = \boldsymbol{\beta}\left(\mathbf{e}_3 + \frac{1}{2}\boldsymbol{\beta}\right), \quad (2.4)$$

$$\boldsymbol{\beta} = \frac{1}{h}(\mathbf{v}^+ - \mathbf{v}^-), \quad \overline{\mathbf{v}} = \frac{1}{2}(\mathbf{v}^- + \mathbf{v}^+), \quad (2.5)$$

where $\overline{H}_\alpha$ are the Lamé parameters of the shell middle surface, $\overline{\delta}$ is the distance from the reference surface to the middle surface, and $\overline{\mathbf{v}}$ is the displacement of the middle surface.

By replacing the Lamé parameters H_α in Eq. (2.2) by the values of these parameters on the face surfaces, $H^\pm_\alpha = A_\alpha(1 + k_\alpha\delta^\pm)$, and in Eq. (2.3) by their values on the middle surface, we arrive at the strain-displacement relationships of the refined geometrically nonlinear theory of thin shells,

$$2\varepsilon^b_{\alpha\beta} = \left[N^-(\alpha_3)\frac{1}{H^-_\alpha}\mathbf{v}^-_{,\alpha} + N^+(\alpha_3)\frac{1}{H^+_\alpha}\mathbf{v}^+_{,\alpha}\right]\mathbf{e}_\beta + \left[N^-(\alpha_3)\frac{1}{H^-_\beta}\mathbf{v}^-_{,\beta} + N^+(\alpha_3)\frac{1}{H^+_\beta}\mathbf{v}^+_{,\beta}\right]\mathbf{e}_\alpha$$
$$+ \left[N^-(\alpha_3)\frac{1}{H^-_\alpha}\mathbf{v}^-_{,\alpha} + N^+(\alpha_3)\frac{1}{H^+_\alpha}\mathbf{v}^+_{,\alpha}\right]\left[N^-(\alpha_3)\frac{1}{H^-_\beta}\mathbf{v}^-_{,\beta} + N^+(\alpha_3)\frac{1}{H^+_\beta}\mathbf{v}^+_{,\beta}\right], \quad (2.6)$$
$$2\varepsilon^b_{\alpha\beta} = \boldsymbol{\beta}\mathbf{e}_\alpha + \frac{1}{\overline{H}_\alpha}\overline{\mathbf{v}}_{,\alpha}(\mathbf{e}_3 + \boldsymbol{\beta}) + (\alpha_3 - \overline{\delta})\frac{1}{\overline{H}_\alpha}\varepsilon^b_{33,\alpha}, \quad \varepsilon^b_{33} = \boldsymbol{\beta}\left(\mathbf{e}_3 + \frac{1}{2}\boldsymbol{\beta}\right),$$

where the vectors $\boldsymbol{\beta}$ and $\overline{\mathbf{v}}$ are defined in (2.5).

Unlike the relationships of (2.2)–(2.4), the relationships of (2.6) are rather attractive for FEM applications, since these relationships provide exact expressions for arbitrarily large rigid displacements of the shell. To prove this, we represent the rigid displacements of the face surfaces of the shell, in accordance with (1.2) and (2.1), in the form

$$\mathbf{v}^{\pm R} = \boldsymbol{\Delta} + (\mathbf{B} - \mathbf{E})\mathbf{R}^\pm, \qquad \mathbf{R}^\pm = \mathbf{r} + \delta^\pm\mathbf{e}_3, \quad (2.7)$$

where $\mathbf{R}^\pm$ are the position vectors of points of the face surfaces $S^\pm$ (Fig. 1). For the derivatives of these vectors, we have the expression similar to that of (1.4),

$$\mathbf{v}^{\pm R}_{,\alpha} = H^\pm_\alpha(\mathbf{B}\mathbf{e}_\alpha - \mathbf{e}_\alpha). \quad (2.8)$$

Substitute the expressions of (2.7) and (2.8) into the strain-displacement relationships of (2.5) and (2.6). With reference to the identity $N^-(\alpha_3) + N^+(\alpha_3) = 1$ and the property of an orthogonal transformation to preserve the scalar product, we obtain

$$2\varepsilon^{bR}_{\alpha\beta} = (\mathbf{B}\mathbf{e}_\alpha)(\mathbf{B}\mathbf{e}_\beta) - \mathbf{e}_\alpha\mathbf{e}_\beta = 0, \quad 2\varepsilon^{bR}_{33} = (\mathbf{B}\mathbf{e}_3)(\mathbf{B}\mathbf{e}_3) - \mathbf{e}_3\mathbf{e}_3 = 0, \quad 2\varepsilon^{bR}_{\alpha 3} = \frac{H_\alpha}{\overline{H}_\alpha}[(\mathbf{B}\mathbf{e}_\alpha)(\mathbf{B}\mathbf{e}_3) - \mathbf{e}_\alpha\mathbf{e}_3] = 0. \quad (2.9)$$

This completes the proof.

A disadvantage of the strain-displacement relationships of (2.6) is that the tangential components of the Green–Lagrange strain tensor change in accordance with a quadratic law along the thickness. This substantially hampers practical applications of these relationships, including FEM applications [3, 5]. More convenient strain-displacement relationships of the geometrically nonlinear shells for Timoshenko-type shells can be written as follows:

$$2\varepsilon^c_{\alpha\beta} = N^-(\alpha_3)\left(\frac{1}{H^-_\alpha}\mathbf{v}^-_{,\alpha}\mathbf{e}_\beta + \frac{1}{H^-_\beta}\mathbf{v}^-_{,\beta}\mathbf{e}_\alpha + \frac{1}{H^-_\alpha H^-_\beta}\mathbf{v}^-_{,\alpha}\mathbf{v}^-_{,\beta}\right)$$
$$+ N^+(\alpha_3)\left(\frac{1}{H^+_\alpha}\mathbf{v}^+_{,\alpha}\mathbf{e}_\beta + \frac{1}{H^+_\beta}\mathbf{v}^+_{,\beta}\mathbf{e}_\alpha + \frac{1}{H^+_\alpha H^+_\beta}\mathbf{v}^+_{,\alpha}\mathbf{v}^+_{,\beta}\right), \tag{2.10}$$
$$2\varepsilon^c_{\alpha 3} = \boldsymbol{\beta}\mathbf{e}_\alpha + \frac{1}{\overline{H}_\alpha}\overline{\mathbf{v}}_{,\alpha}(\mathbf{e}_3 + \boldsymbol{\beta}) + (\alpha_3 - \overline{\delta})\frac{1}{\overline{H}_\alpha}\varepsilon^c_{33,\alpha}, \quad \varepsilon^c_{33} = \boldsymbol{\beta}\left(\mathbf{e}_3 + \frac{1}{2}\boldsymbol{\beta}\right).$$

By substituting the expressions of (2.7) and (2.8) into the relationships of (2.9) one can readily verify that these relationships provide exact expressions for arbitrarily large rigid displacements of the shell, since, as was the case for (2.9), we have $\varepsilon^{cR}_{ij} = 0$.

It is important that the strain-displacement relationships obtained satisfy the conditions

$$\varepsilon^a_{\alpha\beta}(\delta^\pm) = \varepsilon^b_{\alpha\beta}(\delta^\pm) = \varepsilon^c_{\alpha\beta}(\delta^\pm) = E^\pm_{\alpha\beta}, \quad \varepsilon^a_{\alpha 3}(\overline{\delta}) = \varepsilon^b_{\alpha 3}(\overline{\delta}) = \varepsilon^c_{\alpha 3}(\overline{\delta}) = \overline{E}_{\alpha 3}, \tag{2.11}$$

where $E^\pm_{\alpha\beta}$ are the tangential strains of the face surfaces $S^\pm$ and $\overline{E}_{\alpha 3}$ are the transverse shears of the shell middle surface. The geometrical interpretation of the conditions of (2.11) is shown in Fig. 2.

For practical applications of the strain-displacement relationships of (2.10), we represent these relationships in the scalar form

$$\varepsilon^c_{\alpha\beta} = N^-(\alpha_3)E^-_{\alpha\beta} + N^+(\alpha_3)E^+_{\alpha\beta}, \quad \varepsilon^c_{\alpha 3} = N^-(\alpha_3)E^-_{\alpha 3} + N^+(\alpha_3)E^+_{\alpha 3}, \quad \varepsilon^c_{33} = E_{33}, \tag{2.12}$$

where

$$E^\pm_{\alpha\beta} = e^\pm_{\alpha\beta} + \eta^\pm_{\alpha\beta}, \quad E^\pm_{\alpha 3} = e^\pm_{\alpha 3} + \eta^\pm_{\alpha 3}, \quad E_{33} = e_{33} + \eta_{33}, \tag{2.13}$$
$$e^\pm_{\alpha\alpha} = \frac{1}{\zeta^\pm_\alpha}\lambda^\pm_\alpha, \quad 2e^\pm_{12} = \frac{1}{\zeta^\pm_1}\omega^\pm_1 + \frac{1}{\zeta^\pm_2}\omega^\pm_2, \quad 2e^\pm_{\alpha 3} = \frac{\zeta^\pm_\alpha}{\overline{\zeta}_\alpha}\beta_\alpha - \frac{1}{\overline{\zeta}_\alpha}\theta^\pm_\alpha, \quad e_{33} = \beta_3,$$
$$\eta^\pm_{\alpha\alpha} = \frac{1}{2(\zeta^\pm_\alpha)^2}\left[(\lambda^\pm_\alpha)^2 + (\omega^\pm_\alpha)^2 + (\theta^\pm_\alpha)^2\right], \quad 2\eta^\pm_{12} = \frac{1}{\zeta^\pm_1\zeta^\pm_2}(\lambda^\pm_1\omega^\pm_2 + \lambda^\pm_2\omega^\pm_1 + \theta^\pm_1\theta^\pm_2),$$
$$2\eta^\pm_{\alpha 3} = \frac{1}{\overline{\zeta}_\alpha}(\beta_\alpha\lambda^\pm_\alpha + \beta_\gamma\omega^\pm_\alpha - \beta_3\theta^\pm_\alpha), \quad \eta_{33} = \frac{\beta_1^2 + \beta_2^2 + \beta_3^2}{2},$$
$$\lambda^\pm_\alpha = \frac{1}{A_\alpha}v^\pm_{\alpha,\alpha} + B_\gamma v^\pm_\gamma + k_\alpha v^\pm_3, \quad \omega^\pm_\alpha = \frac{1}{A_\alpha}v^\pm_{\gamma,\alpha} - B_\gamma v^\pm_\alpha, \quad \theta^\pm_\alpha = -\frac{1}{A_\alpha}v^\pm_{3,\alpha} + k_\alpha v^\pm_\alpha, \quad \beta_i = \frac{v^+_i - v^-_i}{h},$$
$$\zeta^\pm_\alpha = 1 + k_\alpha\delta^\pm, \quad \overline{\zeta}_\alpha = 1 + k_\alpha\overline{\delta}, \quad B_\alpha = \frac{1}{A_1A_2}A_{\gamma,\alpha} \quad (\gamma \neq \alpha).$$

3. STRAIN-DISPLACEMENT RELATIONSHIPS OF THE GEOMETRICALLY NONLINEAR THEORY OF LAMINATE SHELLS BASED ON THE BROKEN NORMAL HYPOTHESIS

Consider a shell of constant thickness h that consists of N layers each of which has a constant thickness h_k. Here and in what follows, $k = 1, \dots, N$. As the reference surface S we take an internal surface of some, kth, layer or the interface surface between two layers. As was the case previously, we introduce curvilinear orthogonal coordinates α_1 and α_2 on the reference frame measured along the principal curvature lines. The transverse coordinate α_3 is measured along the outward normal to the surface S.

We adopt the kinematical hypothesis according to which the displacements are linearly distributed along the thickness of the kth layer of the shell [15], i.e.,

$$\mathbf{u}^{(k)} = N^-_k(\alpha_3)\mathbf{v}^{(k-1)} + N^+_k(\alpha_3)\mathbf{v}^{(k)}, \qquad \mathbf{v}^{(l)} = \sum_i v^{(l)}_i\mathbf{e}_i, \quad N^-_k(\alpha_3) = \frac{\delta_k - \alpha_3}{h_k}, \quad N^+_k(\alpha_3) = \frac{\alpha_3 - \delta_{k-1}}{h_k}, \tag{3.1}$$

where $\mathbf{v}^{(l)}$ are the displacements of the face surfaces of the layers, $v^{(l)}(\alpha_1, \alpha_2)$ are the components of the vectors $\mathbf{v}^{(l)}$, $N^\pm_k(\alpha_3)$ are linear functions that specify the shape of the kth layer of the shell, and δ_l is the distance from the surface S to the face surface of the kth layer ($l = k - 1$ for the lower surface and $l = k$ for the upper surface).

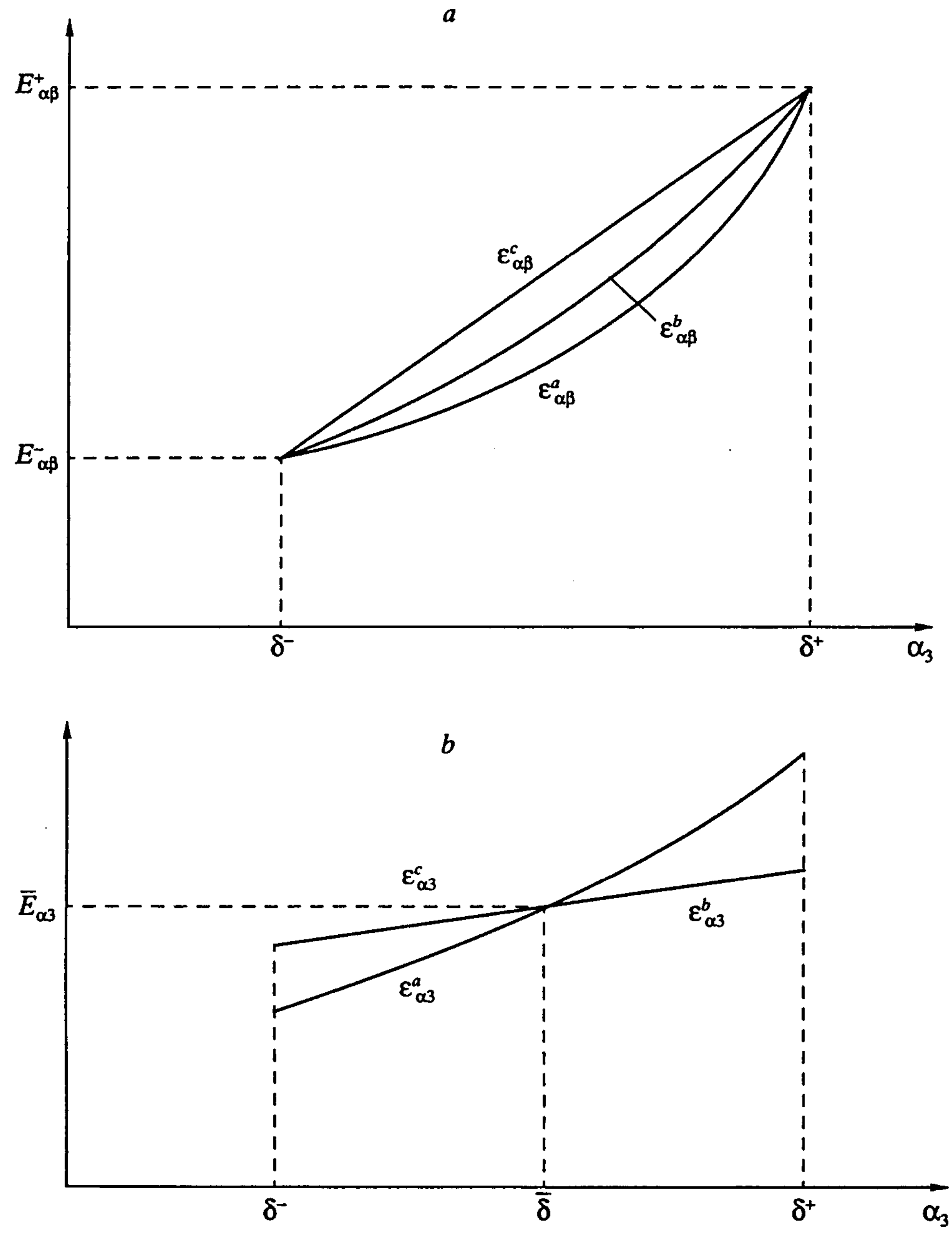

Fig. 2

Substitute the expressions of (3.1) for the displacements into the strain-displacement relationships (1.1) of nonlinear elasticity. Taking into account the differentiation rules for the basis vectors [7], after appropriate transformations we obtain the strain-displacement relationships of the geometrically nonlinear theory of medium-thickness shells,

$$
\begin{aligned}
2\varepsilon^{(k)a}_{\alpha\beta} = {} & \left[N^-_k(\alpha_3)\frac{1}{H_\alpha}\mathbf{v}^{(k-1)}_{,\alpha} + N^+_k(\alpha_3)\frac{1}{H_\alpha}\mathbf{v}^{(k)}_{,\alpha}\right]\mathbf{e}_\beta + \left[N^-_k(\alpha_3)\frac{1}{H_\beta}\mathbf{v}^{(k-1)}_{,\beta} + N^+_k(\alpha_3)\frac{1}{H_\beta}\mathbf{v}^{(k)}_{,\beta}\right]\mathbf{e}_\alpha \\
& + \left[N^-_k(\alpha_3)\frac{1}{H_\alpha}\mathbf{v}^{(k-1)}_{,\alpha} + N^+_k(\alpha_3)\frac{1}{H_\alpha}\mathbf{v}^{(k)}_{,\alpha}\right]\left[N^-_k(\alpha_3)\frac{1}{H_\beta}\mathbf{v}^{(k-1)}_{,\beta} + N^+_k(\alpha_3)\frac{1}{H_\beta}\mathbf{v}^{(k)}_{,\beta}\right],
\end{aligned}
\tag{3.2}
$$

$$
2\varepsilon^{(k)a}_{\alpha 3} = \frac{\overline{H}^{(k)}_\alpha}{H_\alpha}\beta^{(k)}\mathbf{e}_\alpha + \frac{1}{H_\alpha}\overline{\mathbf{v}}^{(k)}_{,\alpha}(\mathbf{e}_3 + \beta^{(k)}) + (\alpha_3 - \overline{\delta}_k)\frac{1}{H_\alpha}\varepsilon^{(k)a}_{33,\alpha},
\tag{3.3}
$$

$$\varepsilon_{33}^{(k)a} = \beta^{(k)}\left(\mathbf{e}_3 + \frac{1}{2}\beta^{(k)}\right), \qquad \overline{H}_\alpha^{(k)} = A_\alpha(1 + k_\alpha\overline{\delta}_k), \quad \overline{\delta}_k = \frac{\delta_{k-1} + \delta_k}{2}, \tag{3.4}$$

$$\beta^{(k)} = \frac{1}{h_k}(\mathbf{v}^{(k)} - \mathbf{v}^{(k-1)}), \quad \overline{\mathbf{v}}^{(k)} = \frac{1}{2}(\mathbf{v}^{(k-1)} + \mathbf{v}^{(k)}), \tag{3.5}$$

where $\overline{H}_\alpha^{(k)}$ are the Lamé parameters of the middle surface of the kth layer, $\overline{\delta}_k$ is the distance from the reference surface S to the middle surface of the kth layer, and $\overline{\mathbf{v}}^{(k)}$ is the displacement of the middle surface of the kth layer.

By changing the Lamé parameters H_α in (3.2) by their values on the face surfaces of the kth layer, $H_\alpha^{(k-1)}$ and $H_\alpha^{(k)}$, and in (3.3) by the values on the middle surface of the kth layer, $\overline{H}_\alpha^{(k)}$, we arrive at the strain-displacement relationships of the geometrically nonlinear theory of thin laminate shells,

$$\begin{aligned}
2\varepsilon_{\alpha\beta}^{(k)b} &= \left[N_k^-(\alpha_3)\frac{1}{H_\alpha^{(k-1)}}\mathbf{v}_{,\alpha}^{(k-1)} + N_k^+(\alpha_3)\frac{1}{H_\alpha^{(k)}}\mathbf{v}_{,\alpha}^{(k)}\right]\mathbf{e}_\beta + \left[N_k^-(\alpha_3)\frac{1}{H_\beta^{(k-1)}}\mathbf{v}_{,\beta}^{(k-1)} + N_k^+(\alpha_3)\frac{1}{H_\beta^{(k)}}\mathbf{v}_{,\beta}^{(k)}\right]\mathbf{e}_\alpha \\
&\quad + \left[N_k^-(\alpha_3)\frac{1}{H_\alpha^{(k-1)}}\mathbf{v}_{,\alpha}^{(k-1)} + N_k^+(\alpha_3)\frac{1}{H_\alpha^{(k)}}\mathbf{v}_{,\alpha}^{(k)}\right]\left[N_k^-(\alpha_3)\frac{1}{H_\beta^{(k-1)}}\mathbf{v}_{,\beta}^{(k-1)} + N_k^+(\alpha_3)\frac{1}{H_\beta^{(k)}}\mathbf{v}_{,\beta}^{(k)}\right], \\
2\varepsilon_{\alpha 3}^{(k)b} &= \beta^{(k)}\mathbf{e}_\alpha + \frac{1}{\overline{H}_\alpha^{(k)}}\overline{\mathbf{v}}_{,\alpha}^{(k)}(\mathbf{e}_3 + \beta^{(k)}) + (\alpha_3 - \overline{\delta}_k)\frac{1}{\overline{H}_\alpha^{(k)}}\varepsilon_{33,\alpha}^{(k)b}, \\
\varepsilon_{33}^{(k)b} &= \beta^{(k)}\left(\mathbf{e}_3 + \frac{1}{2}\beta^{(k)}\right), \qquad \overline{H}_\alpha^{(l)} = A_\alpha(1 + k_\alpha\delta_l) \quad (l = k-1, k).
\end{aligned} \tag{3.6}$$

Unlike (3.2)–(3.4), the relationships of (3.6) give exact expressions for arbitrarily large rigid displacements of the shell layers.

To prove this fundamental proposition, consider the expressions for the displacements of the face surfaces of the kth layer and the derivatives of these displacements with respect to the curvilinear coordinates,

$$\mathbf{v}^{(l)R} = \mathbf{\Delta} + (\mathbf{B} - \mathbf{E})\mathbf{R}^{(l)}, \qquad \mathbf{R}^{(l)} = \mathbf{r} + \delta_l\mathbf{e}_3, \tag{3.7}$$

$$\mathbf{v}_{,\alpha}^{(l)R} = H_\alpha^{(l)}(\mathbf{B}\mathbf{e}_\alpha - \mathbf{e}_\alpha) \qquad (l = k-1, k), \tag{3.8}$$

where $\mathbf{R}^{(l)}$ are the position vectors of the kth layer face surfaces. Substitute (3.7) and (3.8) into the relationships of (3.5) and (3.6). Then, taking into account the identity $N_k^-(\alpha_3) + N_k^+(\alpha_3) = 1$ and the fact that the transformation $\mathbf{B}$ preserves the scalar product, we arrive at the desired relationships

$$2\varepsilon_{\alpha\beta}^{(k)bR} = (\mathbf{B}\mathbf{e}_\alpha)(\mathbf{B}\mathbf{e}_\beta) - \mathbf{e}_\alpha\mathbf{e}_\beta = 0, \quad 2\varepsilon_{33}^{(k)bR} = (\mathbf{B}\mathbf{e}_3)(\mathbf{B}\mathbf{e}_3) - \mathbf{e}_3\mathbf{e}_3 = 0, \quad 2\varepsilon_{\alpha 3}^{(k)bR} = \frac{H_\alpha}{\overline{H}_\alpha^{(k)}}[(\mathbf{B}\mathbf{e}_\alpha)(\mathbf{B}\mathbf{e}_3) - \mathbf{e}_\alpha\mathbf{e}_3] = 0. \tag{3.9}$$

More simple and convenient strain-displacement relationships of the geometrically nonlinear theory of laminate shells can be obtained on the basis if the broken normal hypothesis. To that end, one should ignore the quadratic terms depending on the transverse coordinate α_3 in the expressions of (2.6) for the tangential components of the Green–Lagrange strain tensor. As a result we obtain

$$\begin{aligned}
2\varepsilon_{\alpha\beta}^{(k)c} &= N_k^-(\alpha_3)\left(\frac{1}{H_\alpha^{(k-1)}}\mathbf{v}_{,\alpha}^{(k-1)}\mathbf{e}_\beta + \frac{1}{H_\beta^{(k-1)}}\mathbf{v}_{,\beta}^{(k-1)}\mathbf{e}_\alpha + \frac{1}{H_\alpha^{(k-1)}H_\beta^{(k-1)}}\mathbf{v}_{,\alpha}^{(k-1)}\mathbf{v}_{,\beta}^{(k-1)}\right) \\
&\quad + N_k^+(\alpha_3)\left(\frac{1}{H_\alpha^{(k)}}\mathbf{v}_{,\alpha}^{(k)}\mathbf{e}_\beta + \frac{1}{H_\beta^{(k)}}\mathbf{v}_{,\beta}^{(k)}\mathbf{e}_\alpha + \frac{1}{H_\alpha^{(k)}H_\beta^{(k)}}\mathbf{v}_{,\alpha}^{(k)}\mathbf{v}_{,\beta}^{(k)}\right), \\
2\varepsilon_{\alpha 3}^{(k)c} &= \beta^{(k)}\mathbf{e}_\alpha + \frac{1}{\overline{H}_\alpha^{(k)}}\overline{\mathbf{v}}_{,\alpha}^{(k)}(\mathbf{e}_3 + \beta^{(k)}) + (\alpha_3 - \overline{\delta}_k)\frac{1}{\overline{H}_\alpha^{(k)}}\varepsilon_{33,\alpha}^{(k)c}, \quad \varepsilon_{33}^{(k)c} = \beta^{(k)}\left(\mathbf{e}_3 + \frac{1}{2}\beta^{(k)}\right).
\end{aligned} \tag{3.10}$$

These relationships provide exact expressions for arbitrarily large rigid displacements of the shell kth layer, since $\varepsilon_{ij}^{(k)cR} = 0$, as was the case for (3.9).

The strain-displacement relationships of the theory of laminate shells constructed on the basis of the broken normal hypothesis satisfy the conditions

$$\varepsilon_{\alpha\beta}^{(k)a}(\delta_l) = \varepsilon_{\alpha\beta}^{(k)b}(\delta_l) = \varepsilon_{\alpha\beta}^{(k)c}(\delta_l) = E_{\alpha\beta}^{(l)} \quad (l = k-1, k), \qquad \varepsilon_{\alpha 3}^{(k)a}(\overline{\delta}_k) = \varepsilon_{\alpha 3}^{(k)b}(\overline{\delta}_k) = \varepsilon_{\alpha 3}^{(k)c}(\overline{\delta}_k) = \overline{E}_{\alpha 3}^{(k)}, \tag{3.11}$$

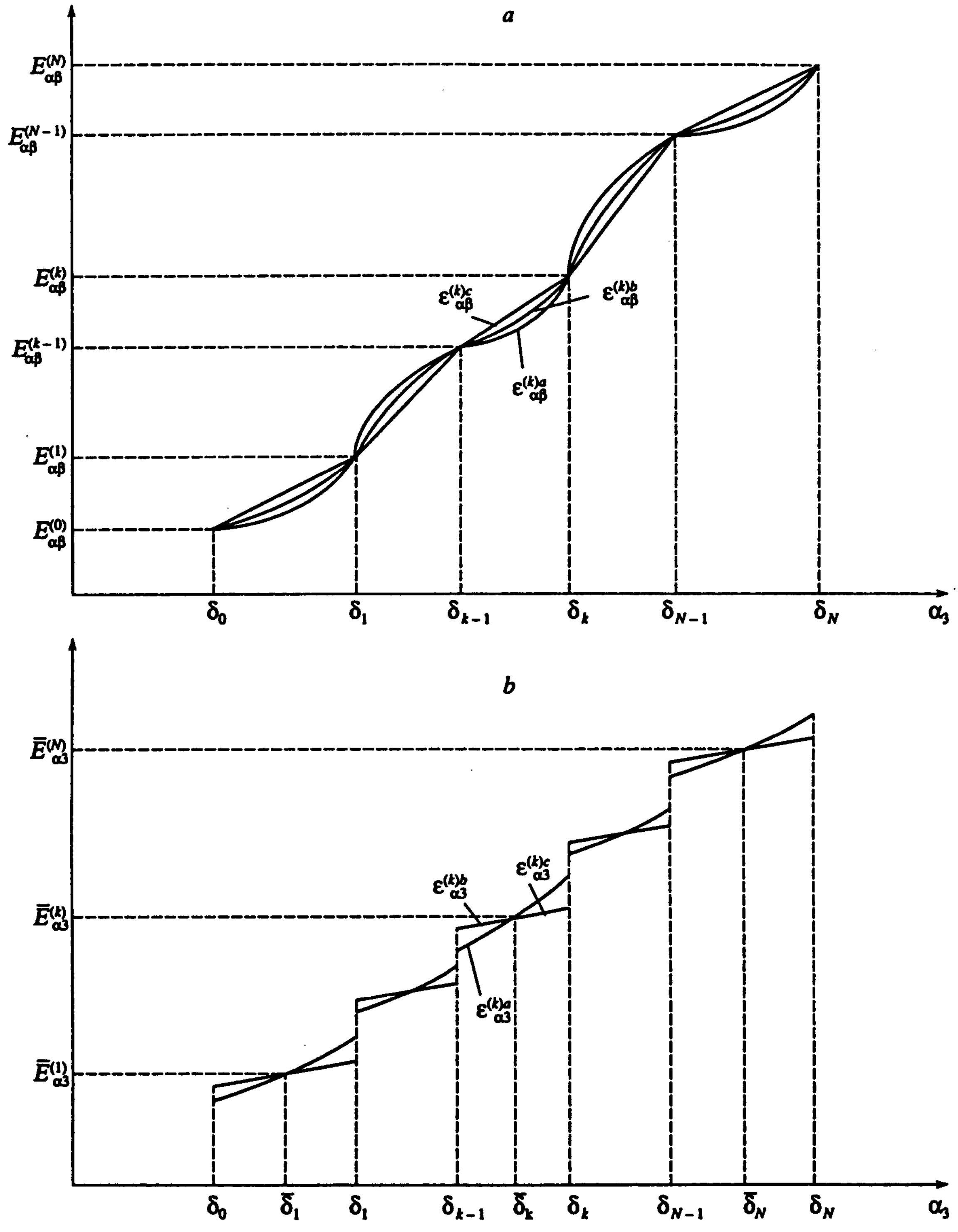

Fig. 3

where $E_{\alpha\beta}^{(k)}$ are the tangential stresses on the kth layer face surfaces and $\overline{E}_{\alpha 3}^{(k)}$ are the transverse shears of the middle surface of the kth layer. The geometrical interpretation of the conditions of (3.11) is shown in Fig. 3.

The scalar form of the relationships of (3.10) is given by

$$\varepsilon_{\alpha\beta}^{(k)c} = N_k^-(\alpha_3)E_{\alpha\beta}^{(k-1)} + N_k^+(\alpha_3)E_{\alpha\beta}^{(k)}, \quad \varepsilon_{\alpha 3}^{(k)c} = N_k^-(\alpha_3)E_{\alpha 3}^{(k)-} + N_k^+(\alpha_3)E_{\alpha 3}^{(k)+}, \quad \varepsilon_{33}^{(k)c} = E_{33}^{(k)}, \tag{3.12}$$

$$
\begin{gathered}
E^{(l)}_{\alpha\beta} = e^{(l)}_{\alpha\beta} + \eta^{(l)}_{\alpha\beta}, \quad E^{(k)\pm}_{\alpha 3} = e^{(k)\pm}_{\alpha 3} + \eta^{(k)\pm}_{\alpha 3}, \quad E^{(k)}_{33} = e^{(k)}_{33} + \eta^{(k)}_{33}, \\
e^{(l)}_{\alpha\alpha} = \frac{1}{\zeta^{(l)}_\alpha}\lambda^{(l)}_\alpha, \quad 2e^{(l)}_{12} = \frac{1}{\zeta^{(l)}_1}\omega^{(l)}_1 + \frac{1}{\zeta^{(l)}_2}\omega^{(l)}_2, \quad e^{(k)}_{33} = \beta^{(k)}_3, \\
2e^{(k)-}_{\alpha 3} = \frac{\zeta^{(k-1)}_\alpha}{\bar{\zeta}^{(k)}_\alpha}\beta^{(k)}_\alpha - \frac{1}{\bar{\zeta}^{(k)}_\alpha}\theta^{(k-1)}_\alpha, \quad 2e^{(k)+}_{\alpha 3} = \frac{\zeta^{(k)}_\alpha}{\bar{\zeta}^{(k)}_\alpha}\beta^{(k)}_\alpha - \frac{1}{\bar{\zeta}^{(k)}_\alpha}\theta^{(k)}_\alpha, \\
\eta^{(l)}_{\alpha\alpha} = \frac{1}{2(\zeta^{(l)}_\alpha)^2}\left[(\lambda^{(l)}_\alpha)^2 + (\omega^{(l)}_\alpha)^2 + (\theta^{(l)}_\alpha)^2\right], \quad 2\eta^{(l)}_{12} = \frac{1}{\zeta^{(l)}_1\zeta^{(l)}_2}\left(\lambda^{(l)}_1\omega^{(l)}_2 + \lambda^{(l)}_2\omega^{(l)}_1 + \theta^{(l)}_1\theta^{(l)}_2\right), \\
2\eta^{(k)-}_{\alpha 3} = \frac{1}{\bar{\zeta}^{(k)}_\alpha}\left(\beta^{(k)}_\alpha\lambda^{(k-1)}_\alpha + \beta^{(k)}_\gamma\omega^{(k-1)}_\alpha - \beta^{(k)}_3\theta^{(k-1)}_\alpha\right), \quad 2\eta^{(k)+}_{\alpha 3} = \frac{1}{\bar{\zeta}^{(k)}_\alpha}\left(\beta^{(k)}_\alpha\lambda^{(k)}_\alpha + \beta^{(k)}_\gamma\omega^{(k)}_\alpha - \beta^{(k)}_3\theta^{(k)}_\alpha\right), \\
\eta_{33} = \frac{(\beta^{(k)}_1)^2 + (\beta^{(k)}_2)^2 + (\beta^{(k)}_3)^2}{2}, \quad \lambda^{(l)}_\alpha = \frac{1}{A_\alpha}v^{(l)}_{\alpha,\alpha} + B_\gamma v^{(l)}_\gamma + k_\alpha v^{(l)}_3, \\
\omega^{(l)}_\alpha = \frac{1}{A_\alpha}v^{(l)}_{\gamma,\alpha} - B_\gamma v^{(l)}_\alpha, \quad \theta^{(l)}_\alpha = -\frac{1}{A_\alpha}v^{(l)}_{3,\alpha} + k_\alpha v^{(l)}_\alpha, \quad \beta^{(k)}_i = \frac{v^{(k)}_i - v^{(k-1)}_i}{h_k}, \\
\zeta^{(l)}_\alpha = 1 + k_\alpha\delta_l, \quad \bar{\zeta}^{(k)}_\alpha = 1 + k_\alpha\bar{\delta}_k, \quad B_\alpha = \frac{1}{A_1A_2}A_{\gamma,\alpha} \qquad (l = k-1, k; \quad \gamma \neq \alpha).
\end{gathered}
\tag{3.13}
$$

4. CONCLUSIONS

In the present paper, the expressions for the Green–Lagrange strain tensor have been constructed in curvilinear coordinates for homogeneous shells on the basis of Timoshenko's kinematic hypothesis ((2.12) and (2.13)) and for laminate shells on the basis of the broken normal hypothesis ((3.12) and (3.13)). These expressions can be successfully utilized in finite element technology for developing new effective finite elements for homogeneous and laminate shells, since these expressions exactly represent arbitrarily large rigid displacements of the finite elements.

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