STRAIN-DISPLACEMENT RELATIONSHIPS THAT EXACTLY REPRESENT LARGE RIGID DISPLACEMENTS OF A SHELL

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The finite element method (FEM) has become the most powerful numerical method that is utilized currently for solving problems in mechanics of shells. At the same time, the problem of the construction of curved finite elements for thin shells subject to large displacements and arbitrarily large rotations is still far from being solved [1-3]. It is for this reason that the representation provided by the strain-displacement relationships for rigid displacements of the shell element is inadequate. Therefore, it is no wonder that since the strain relationships that would be able to represent an arbitrary strain state in local curvilinear coordinates are not available in the literature, the degenerate (isoparametric) element concept [4] has been mostly developed. This concept enables one to represent rigid displacements of the element in the global Cartesian reference frame but requires considerable increase in the computational time [3, 5].

In the literature, there are a lot of versions of both the classical (Kirchhoff–Love) theory of shells and the theory refined on the basis of Timoshenko’s hypothesis. However, only some of these theories (e.g., those of [5–9]) provide adequate approximations for rigid displacements of the shell. This observation is validated in [7, 10, 11] for the geometrically linear Kirchhoff–Love theory of shells and in [12] for the refined theory of shells. In the present paper, the results of [12] are extended to shells subject to large displacements and arbitrarily large rotations.

1. STRAIN-DISPLACEMENT RELATIONSHIPS OF GEOMETRICALLY LINEAR ELASTICITY

Consider a shell of constant thickness  \( h \). As the reference surface \( S \) we take an internal surface of the shell. We introduce on the reference surface the curvilinear coordinates \( \alpha_1 \) and \( \alpha_2 \) measured along the principal curvatures. The transverse coordinate \( \alpha_3 \) is measured along the outward normal to the surface \( S \) (Fig. 1). Let \( e_1 \) and \( e_2 \) denote the unit vectors tangent to the coordinate lines \( \alpha_1 \) and \( \alpha_2 \), \( e_3 \) the unit vector of the outward normal, \( A_\alpha \) and \( k_\alpha \) the Lamé parameters and the curvatures of the coordinate lines of the surface \( S \), \( \delta^s \) and \( \delta^* \) the distances from the surface \( S \) to the lower, \( S^- \), and upper, \( S^+ \), surfaces of the shell, respectively, and \( \varepsilon_{ij} \) are the components of the Green–Lagrange strain tensor. Here and in what follows, Latin indices \( i \) and \( j \) take on the values 1, 2 or 3, and Greek indices \( \alpha, \beta, \) and \( \gamma \) the values 1 or 2.

We will represent the strain-displacement relationships of the geometrically nonlinear theory of elasticity [13] in the vector form

\[
2\varepsilon_{ij} = \frac{1}{H_i} u_i \left( e_j + \frac{1}{2H_j} u_j \right) + \frac{1}{H_j} u_j \left( e_i + \frac{1}{2H_i} u_i \right), \quad u = \sum_i u_i e_i, \quad H_\alpha = A_\alpha (1 + k_\alpha \alpha_3), \quad H_3 = 1, \quad (i, j)
\]

where \( u_i (\alpha_1, \alpha_2, \alpha_3) \) are the displacement vector components, and \( H_\alpha \) are the Lamé parameters of a surface all point of which are equidistant from the reference surface \( S \). (Such surfaces sometimes are referred to as parallel surfaces.) The superscript \( i \) after the comma stands for partial differentiation with respect to the coordinate \( \alpha_i \).

Arbitrarily large rigid displacements of the shell will be also represented in the vector form

\[
\mathbf{u}^R = \Delta + (\mathbf{B} - \mathbf{E}) \mathbf{R}, \quad \mathbf{R} = r + \alpha_3 \mathbf{e}_3, \quad \Delta = \sum_i \Delta_i \mathbf{e}_i,
\]

(1.2)
where \( \mathbf{R} \) is the position vector of a point of the shell, \( \mathbf{r}(\alpha_1, \alpha_2) \) is the position vector of the orthogonal projection of this point onto the reference surface \( S \), \( \Delta \) is the translational displacement of the shell, \( \mathbf{E} \) is the unit matrix, \( \mathbf{B} \) is the orthogonal rotation matrix, and \( \varphi, \theta, \) and \( \psi \) are the Euler–Krylov angles that characterize the rigid rotation of the shell about the point \( O \) (Fig. 1).

The differentiation of (1.2) with respect to \( \alpha_i \) with reference to (1.3) and the relations \( \Delta_i = 0 \) and \( \mathbf{R}_i = H_i e_i [7] \) yields

\[
\mathbf{u}_{i, ij}^R = H_i (\mathbf{B} e_i - e_i). \quad (1.4)
\]

By substituting the derivatives of (1.4) into the strain-displacement relationships of (1.1) we obtain (after simple transformations)

\[
2 \varepsilon_{ij}^R = (\mathbf{B} e_i)(\mathbf{B} e_j) - e_i e_j. \quad (1.5)
\]

With reference to the fact that an orthogonal transformation preserves the scalar product, the relation of (1.5) implies

\[
\varepsilon_{ij}^R = 0. \quad (1.6)
\]

Hence, as could be anticipated, all components of the Green–Lagrange strain tensor in the local curvilinear coordinates \( \alpha_i \) are equal to zero for arbitrarily large rigid displacements of the shell.

### 2. STRAIN-DISPLACEMENT RELATIONSHIPS FOR THE GEOMETRICALLY NONLINEAR Refined THEORY OF SHELLS

We will make use of Timoshenko’s modified kinematic hypothesis according to which the displacements are linearly distributed across the shell thickness [9], specifically,

\[
\mathbf{u} = N^-(\alpha_3) \mathbf{v}^- + N^+(\alpha_3) \mathbf{v}^+, \quad \mathbf{v}^\pm = \sum_i v_i^\pm \mathbf{e}_i, \quad N^-(\alpha_3) = \frac{\delta^+ - \alpha_3}{h}, \quad N^+(\alpha_3) = \frac{\alpha_3 - \delta^-}{h}, \quad (2.1)
\]

where \( \mathbf{v}^\pm \) are the displacement vectors of the face surfaces of the shell, \( S^\pm \), \( v_i^\pm(\alpha_1, \alpha_2) \) are the components of these vectors, and \( N^\pm(\alpha_3) \) are linear functions that specify the shape of the shell.
Substitute the expressions of (2.1) for displacements into the strain-displacement relationships of (1.1) of the spatial theory of elasticity. With reference to the formulas of [7] for differentiation of the basis vectors, $e_{3,\alpha} = A_{\alpha}k_{\alpha}e_{\alpha}$, we arrive at the strain-displacement relationships of the refined geometrically nonlinear theory of medium-thickness shells,

$$
2\varepsilon^{a}_{\alpha\delta} = \left[ N^{a}(\alpha_{3}) \frac{1}{H_{\alpha}}v_{\alpha}^{a} + N^{a*}(\alpha_{3}) \frac{1}{H_{\alpha}}v_{\alpha}^{a*} \right] e_{\delta} + \left[ N^{a}(\alpha_{3}) \frac{1}{H_{\beta}}v_{\alpha}^{a} + N^{a*}(\alpha_{3}) \frac{1}{H_{\beta}}v_{\alpha}^{a*} \right] e_{\alpha},
$$

$$
+ \left[ N^{a}(\alpha_{3}) \frac{1}{H_{\alpha}}v_{\alpha}^{a} + N^{a*}(\alpha_{3}) \frac{1}{H_{\alpha}}v_{\alpha}^{a*} \right] \left[ N^{a}(\alpha_{3}) \frac{1}{H_{\beta}}v_{\alpha}^{a} + N^{a*}(\alpha_{3}) \frac{1}{H_{\beta}}v_{\alpha}^{a*} \right],
$$

(2.2)

$$
2\varepsilon^{2}_{\alpha\delta} = \overline{H}_{\alpha} \beta e_{\alpha} + \frac{1}{H_{\alpha}}\overline{v}_{\alpha}(e_{3} + \delta) \frac{1}{H_{\alpha}}e^{a}_{33,\alpha},
$$

$$
\overline{H}_{\alpha} = A_{\alpha}(1 + k_{\alpha}\delta), \quad \overline{v} = \frac{1}{2}(v^{+} + v^{-}),
$$

(2.3)

$$
\beta = \frac{1}{h}(v^{+} - v^{-}),
$$

(2.4)

$$
\alpha = \frac{1}{2}(v^{+} + v^{-}),
$$

(2.5)

where $H_{\alpha}$ are the Lamé parameters of the shell middle surface, $\delta$ is the distance from the reference surface to the middle surface, and $\overline{v}$ is the displacement of the middle surface.

By replacing the Lamé parameters $H_{\alpha}$ in Eq. (2.2) by the values of these parameters on the face surfaces, $H_{\alpha}^{s} = A_{\alpha}(1 + k_{\alpha}\delta^{s})$, and in Eq. (2.3) by their values on the middle surface, we arrive at the strain-displacement relationships of the refined geometrically nonlinear theory of thin shells,

$$
2\varepsilon^{b}_{\alpha\delta} = \left[ N^{b}(\alpha_{3}) \frac{1}{H_{\alpha}}v_{\alpha}^{b} + N^{b*}(\alpha_{3}) \frac{1}{H_{\alpha}}v_{\alpha}^{b*} \right] e_{\delta} + \left[ N^{b}(\alpha_{3}) \frac{1}{H_{\beta}}v_{\alpha}^{b} + N^{b*}(\alpha_{3}) \frac{1}{H_{\beta}}v_{\alpha}^{b*} \right] e_{\alpha},
$$

$$
+ \left[ N^{b}(\alpha_{3}) \frac{1}{H_{\alpha}}v_{\alpha}^{b} + N^{b*}(\alpha_{3}) \frac{1}{H_{\alpha}}v_{\alpha}^{b*} \right] \left[ N^{b}(\alpha_{3}) \frac{1}{H_{\beta}}v_{\alpha}^{b} + N^{b*}(\alpha_{3}) \frac{1}{H_{\beta}}v_{\alpha}^{b*} \right],
$$

(2.6)

$$
2\varepsilon^{b}_{\alpha\delta} = \beta e_{\alpha} + \frac{1}{H_{\alpha}}\overline{v}_{\alpha}(e_{3} + \delta) \frac{1}{H_{\alpha}}e^{b}_{33,\alpha},
$$

$$
e^{b}_{33} = \beta \left( e_{3} + \frac{1}{2} \beta \right),
$$

where the vectors $\beta$ and $\overline{v}$ are defined in (2.5).

Unlike the relationships of (2.2)–(2.4), the relationships of (2.6) are rather attractive for FEM applications, since these relationships provide exact expressions for arbitrarily large rigid displacements of the shell. To prove this, we represent the rigid displacements of the face surfaces of the shell, in accordance with (1.2) and (2.1), in the form

$$
v^{R} = \Delta + (B - E)R^{R}, \quad R^{R} = r + \delta^{R}e_{3},
$$

(2.7)

where $R^{R}$ are the position vectors of points of the face surfaces $S^{R}$ (Fig. 1). For the derivatives of these vectors, we have the expression similar to that of (1.4),

$$
v^{a}_{33} = H^{a}_{3}e_{a}. \quad e_{a} = (B_{a} - e_{a}),
$$

(2.8)

Substitute the expressions of (2.7) and (2.8) into the strain-displacement relationships of (2.5) and (2.6). With reference to the identity $N^{a}(\alpha_{3}) + N^{a*}(\alpha_{3}) = 1$ and the property of an orthogonal transformation to preserve the scalar product, we obtain

$$
2\varepsilon^{b}_{a\delta} = (B_{a})(B_{b}) - e_{a}e_{b} = 0, \quad 2\varepsilon^{b}_{33} = (B_{3})(B_{3}) - e_{3}e_{3} = 0, \quad 2\varepsilon^{b}_{a3} = H^{a}_{3}(B_{a})(B_{3}) - e_{a}e_{3} = 0.
$$

(2.9)

This completes the proof.

A disadvantage of the strain-displacement relationships of (2.6) is that the tangential components of the Green–Lagrange strain tensor change in accordance with a quadratic law along the thickness. This substantially hampers practical applications of these relationships, including FEM applications [3, 5]. More convenient strain-displacement
relationships of the geometrically nonlinear shells for Timoshenko-type shells can be written as follows:

\[
2\varepsilon_{\alpha\beta}^c = N^{-}(\alpha_3) \left( \frac{1}{H_\alpha} \nabla_{\alpha\beta} \varepsilon_{\alpha} + \frac{1}{H_\beta} \nabla_{\alpha\beta} \varepsilon_{\beta} + \frac{1}{H_\alpha H_\beta} \nabla_{\alpha} \nabla_{\beta} \right) + N^{+}(\alpha_3) \left( \frac{1}{H_\alpha} \nabla_{\alpha\beta} \varepsilon_{\alpha} + \frac{1}{H_\beta} \nabla_{\beta} \varepsilon_{\alpha} + \frac{1}{H_\alpha H_\beta} \nabla_{\alpha} \nabla_{\beta} \right),
\]

\[
2\varepsilon_{\alpha_3}^c = \beta \varepsilon_{\alpha} + \frac{1}{H_\alpha} \nabla_{\alpha} (\varepsilon_{3} + \beta) + (\alpha_3 - \delta) \frac{1}{H_\alpha} \varepsilon_{\alpha_3}, \quad \varepsilon_{\alpha_3}^c = \beta (\varepsilon_{3} + \frac{1}{2} \beta).
\]

(2.10)

By substituting the expressions of (2.7) and (2.8) into the relationships of (2.9) one can readily verify that these relationships provide exact expressions for arbitrarily large rigid displacements of the shell, since, as was the case for (2.9), we have \( \varepsilon_{\alpha_3}^R = 0 \).

It is important that the strain-displacement relationships obtained satisfy the conditions

\[
\varepsilon_{\alpha_3}^c (\delta^\pm) = \varepsilon_{\alpha_3}^b (\delta^\pm) = \varepsilon_{\alpha_3}^c (\delta^\pm) = E_{\alpha_3}^{\pm c}, \quad \varepsilon_{\alpha_3}^b (\delta) = \varepsilon_{\alpha_3}^c (\delta) = E_{\alpha_3}^c = \nabla_{\alpha_3},
\]

(2.11)

where \( E_{\alpha_3}^{\pm c} \) are the tangential strains of the face surfaces \( S^\pm \) and \( E_{\alpha_3} \) are the transverse shears of the shell middle surface. The geometrical interpretation of the conditions of (2.11) is shown in Fig. 2.

For practical applications of the strain-displacement relationships of (2.10), we represent these relationships in the scalar form

\[
2\varepsilon_{\alpha_3}^c = N^{-}(\alpha_3) E_{\alpha_3} - N^{+}(\alpha_3) E_{\alpha_3}^+ + N^{-}(\alpha_3) E_{\alpha_3}^- + N^{+}(\alpha_3) E_{\alpha_3}^+, \quad \varepsilon_{\alpha_3}^c = E_{\alpha_3}^- + E_{\alpha_3}^+, \quad \varepsilon_{\alpha_3}^c = E_{\alpha_3},
\]

(2.12)

where

\[
E_{\alpha_3}^- = \varepsilon_{\alpha_3}^- + \varepsilon_{\alpha_3}^b, \quad E_{\alpha_3}^+ = \varepsilon_{\alpha_3}^+ + \varepsilon_{\alpha_3}^c, \quad E_{\alpha_3} = \varepsilon_{\alpha_3} + \varepsilon_{\alpha_3}^c, \quad \varepsilon_{\alpha_3} = \varepsilon_{\alpha_3}^c + \varepsilon_{\alpha_3}^b + \varepsilon_{\alpha_3}^c
\]

\[
\varepsilon_{\alpha_3}^c = \frac{\xi_3}{\xi_3^2} \lambda_{\alpha_3}, \quad 2\varepsilon_{\alpha_3}^b = \frac{\xi_3}{\xi_3^2} \omega_{\alpha_3} + \frac{1}{\xi_3} \theta_{\alpha_3}, \quad 2\varepsilon_{\alpha_3} = \frac{\xi_3}{\xi_3^2} \beta_{\alpha_3} - \frac{1}{\xi_3} \theta_{\alpha_3}, \quad \varepsilon_{\alpha_3} = \beta_{\alpha_3},
\]

\[
\eta_{\alpha_3}^c = \frac{1}{2(\xi_3^2)} \left[ (\lambda_{\alpha_3}^c)^2 + (\omega_{\alpha_3}^c)^2 + (\theta_{\alpha_3}^c)^2 \right], \quad 2\eta_{\alpha_3}^b = \frac{1}{\xi_3^2} \xi_3 \left( \lambda_{\alpha_3}^c \theta_{\alpha_3}^c + \lambda_{\alpha_3}^b \omega_{\alpha_3}^c + \beta_{\alpha_3} \omega_{\alpha_3}^c \right),
\]

\[
2\eta_{\alpha_3} = \frac{\beta_{\alpha_3}^2 + \beta_{\alpha_3}^2 + \beta_{\alpha_3}^2}{2}, \quad \eta_{\alpha_3} = \frac{\beta_{\alpha_3}^2 + \beta_{\alpha_3}^2 + \beta_{\alpha_3}^2}{2}, \quad \eta_{\alpha_3} = \frac{\beta_{\alpha_3}^2 + \beta_{\alpha_3}^2 + \beta_{\alpha_3}^2}{2}, \quad \eta_{\alpha_3} = \frac{\beta_{\alpha_3}^2 + \beta_{\alpha_3}^2 + \beta_{\alpha_3}^2}{2},
\]

\[
\lambda_{\alpha_3} = \frac{1}{A_{\alpha_3}} \omega_{\alpha_3} + B_{\alpha_3} \varepsilon_{\alpha_3} + k_{\alpha_3} \omega_{\alpha_3}, \quad \omega_{\alpha_3} = \frac{1}{A_{\alpha_3}} \omega_{\alpha_3} - B_{\alpha_3} \varepsilon_{\alpha_3} + \frac{1}{A_{\alpha_3}} \omega_{\alpha_3} + k_{\alpha_3} \omega_{\alpha_3}, \quad \theta_{\alpha_3} = \frac{1}{A_{\alpha_3}} \omega_{\alpha_3} + \frac{1}{A_{\alpha_3}} \omega_{\alpha_3}, \quad \beta_{\alpha_3} = \frac{1}{A_{\alpha_3}} \omega_{\alpha_3} + k_{\alpha_3} \omega_{\alpha_3}, \quad \beta_{\alpha_3} = \frac{1}{A_{\alpha_3}} \omega_{\alpha_3} + k_{\alpha_3} \omega_{\alpha_3}, \quad \beta_{\alpha_3} = \frac{1}{A_{\alpha_3}} \omega_{\alpha_3} + k_{\alpha_3} \omega_{\alpha_3}, \quad \beta_{\alpha_3} = \frac{1}{A_{\alpha_3}} \omega_{\alpha_3} + k_{\alpha_3} \omega_{\alpha_3},
\]

\[
\gamma_{\alpha_3} = 1 + k_{\alpha_3} \delta_{\alpha_3}, \quad \gamma_{\alpha_3} = 1 + k_{\alpha_3} \delta_{\alpha_3}, \quad B_{\alpha_3} = \frac{1}{A_{\alpha_3} A_{\alpha_3}} A_{\gamma_{\alpha_3}} (\gamma = \alpha).
\]

3. STRAIN-DISPLACEMENT RELATIONSHIPS OF THE GEOMETRICALLY NONLINEAR THEORY OF LAMINATE SHELLS BASED ON THE BROKEN NORMAL HYPOTHESIS

Consider a shell of constant thickness \( h \) that consists of \( N \) layers each of which has a constant thickness \( h_k \). Here and in what follows, \( k = 1, \ldots, N \). As the reference surface \( S \) we take an internal surface of some, \( k \)-th, layer or the interface surface between two layers. As was the case previously, we introduce curvilinear orthogonal coordinates \( \alpha_1 \) and \( \alpha_2 \) on the reference frame measured along the principal curvature lines. The transverse coordinate \( \alpha_3 \) is measured along the outward normal to the surface \( S \).

We adopt the kinematical hypothesis according to which the displacements are linearly distributed along the thickness of the \( k \)-th layer of the shell [15], i.e.,

\[
u^{(k)} = N_k^{(k)}(\alpha_3) \nu^{(k-1)} + N_k^{(k)}(\alpha_3) \nu^{(k)}, \quad \nu^{(k)} = \sum_i v_i^{(k)} e_i, \quad N_k^{(k)}(\alpha_3) = \frac{\delta_k - \alpha_3}{h_k}, \quad N_k^{(k)}(\alpha_3) = \frac{\alpha_3 - \delta_{k-1}}{h_k}, \]

(3.1)

where \( \nu^{(k)} \) are the displacements of the face surfaces of the layers, \( \nu^{(k)}(\alpha_1, \alpha_2) \) are the components of the vectors \( \nu^{(k)} \), \( N_k^{(k)}(\alpha_3) \) are linear functions that specify the shape of the \( k \)-th layer of the shell, and \( \delta_i \) is the distance from the surface \( S \) to the face surface of the \( k \)-th layer (\( i = k - 1 \) for the lower surface and \( i = k \) for the upper surface).
Substitute the expressions of (3.1) for the displacements into the strain-displacement relationships (1.1) of nonlinear elasticity. Taking into account the differentiation rules for the basis vectors [7], after appropriate transformations we obtain the strain-displacement relationships of the geometrically nonlinear theory of medium-thickness shells,

\[
2e^{(k)\alpha}_\beta = \left[ N^+_k(\alpha_3) \frac{1}{H_\alpha} \psi^{(k-1)}(e_3) + N^-_k(\alpha_3) \frac{1}{H_\alpha} \psi^{(k)}(e_3) \right] e_\beta + \left[ N^+_k(\alpha_3) \frac{1}{H_\beta} \psi^{(k-1)}(e_3) + N^-_k(\alpha_3) \frac{1}{H_\beta} \psi^{(k)}(e_3) \right] e_\alpha
\]

\[
+ \left[ N^+_k(\alpha_3) \frac{1}{H_\alpha} \psi^{(k-1)}(e_3) + N^-_k(\alpha_3) \frac{1}{H_\alpha} \psi^{(k)}(e_3) \right] \left[ N^+_k(\alpha_3) \frac{1}{H_\beta} \psi^{(k-1)}(e_3) + N^-_k(\alpha_3) \frac{1}{H_\beta} \psi^{(k)}(e_3) \right],
\] (3.2)

\[
2e^{(k)\alpha}_3 = \frac{H^{(k)}_\alpha}{H_\alpha} \beta^{(k)} e_\alpha + \frac{1}{H_\alpha} \psi^{(k)} (e_3) (e_3 + \beta^{(k)}) (\alpha_3 - \bar{\alpha}_k) \frac{1}{H_\alpha} e^{(k)\alpha}_3,
\] (3.3)
\[ \varepsilon^{(k\alpha)}_{33} = \beta^{(k)} \left( \varepsilon_{3} + \frac{1}{2} \beta^{(k)} \right), \quad H^{(k)}_{\alpha} = A_{\alpha} \left( 1 + k_{\alpha} \delta_{k} \right), \quad \delta_{k} = \frac{\delta_{k-1} + \delta_{k}}{2}, \] (3.4)

\[ \beta^{(k)} = \frac{1}{k_{\alpha}} \left( \varepsilon^{(k-1)}_{3} - \varepsilon^{(k-1)}_{3} \right), \quad \delta^{(k)} = \frac{1}{2} \left( \varepsilon^{(k-1)}_{3} + \varepsilon^{(k)}_{3} \right), \] (3.5)

where \( H^{(k)}_{\alpha} \) are the Lamé parameters of the middle surface of the \( k \)th layer, \( \delta_{k} \) is the distance from the reference surface \( S \) to the middle surface of the \( k \)th layer, and \( \delta^{(k)} \) is the displacement of the middle surface of the \( k \)th layer.

By changing the Lamé parameters \( H_{\alpha} \) in (3.2) by their values on the face surfaces of the \( k \)th layer, \( H^{(k-1)}_{\alpha} \) and \( H^{(k)}_{\alpha} \), and in (3.3) by the values on the middle surface of the \( k \)th layer, \( H^{(k)}_{\alpha} \), we arrive at the strain-displacement relationships of the geometrically nonlinear theory of thin laminate shells,

\[ 2\varepsilon^{(k\beta)}_{\alpha \beta} = \left[ N^{(k)}_{\alpha}(\alpha_{3}) - \frac{1}{H^{(k-1)}_{\alpha}} \varepsilon^{(k-1)}_{\alpha} \right] e_{\alpha} + \left[ N^{(k)}_{\beta}(\alpha_{3}) \right] \frac{1}{H^{(k-1)}_{\beta}} \varepsilon^{(k-1)}_{\beta} \right] e_{\beta} + \left[ N^{(k)}_{\alpha}(\alpha_{3}) \right] \frac{1}{H^{(k)}_{\alpha}} \varepsilon^{(k)}_{\alpha} \right] e_{\alpha} \]

\[ + \left[ N^{(k)}_{\beta}(\alpha_{3}) \right] \frac{1}{H^{(k)}_{\beta}} \varepsilon^{(k)}_{\beta} \right] e_{\beta} \]

\[ 2\varepsilon^{(k\alpha)}_{\alpha \beta} = \beta^{(k)} e_{\alpha} + \frac{1}{H^{(k)}_{\alpha}} \varepsilon^{(k)}_{\alpha} (e_{3} + \beta^{(k)}) + (\alpha_{3} - \delta_{k}) \frac{1}{H^{(k)}_{\alpha}} \varepsilon^{(k\alpha)}_{33}, \]

\[ \varepsilon^{(k\beta)}_{33} = \beta^{(k)} \left( \varepsilon_{3} + \frac{1}{2} \beta^{(k)} \right), \quad H^{(l)}_{\alpha} = A_{\alpha} \left( 1 + k_{\alpha} \delta_{l} \right) \quad (l = k - 1, k). \] (3.6)

Unlike (3.2)–(3.4), the relationships of (3.6) give exact expressions for arbitrarily large rigid displacements of the shell layers.

To prove this fundamental proposition, consider the expressions for the displacements of the face surfaces of the \( k \)th layer and the derivatives of these displacements with respect to the curvilinear coordinates,

\[ \varepsilon^{(l)} = A + (B - E)R^{(l)}, \quad R^{(l)} = \varepsilon^{(l)} + \delta_{l}, \]

\[ \varepsilon^{(l)} = H^{(l)}_{\alpha} (B_{\alpha} - \varepsilon_{\alpha}), \quad (l = k - 1, k) \] (3.7)

where \( R^{(l)} \) are the position vectors of the \( l \)th layer face surfaces. Substitute (3.7) and (3.8) into the relationships of (3.5) and (3.6). Then, taking into account the identity \( N^{(k)}_{\alpha}(\alpha_{3}) + N^{(k)}_{\beta}(\alpha_{3}) = 1 \) and the fact that the transformation \( B \) preserves the scalar product, we arrive at the desired relationships

\[ 2\varepsilon_{\alpha \beta}^{(k) R} = (B_{\alpha})(B_{\beta}) - \varepsilon_{\alpha} \varepsilon_{\beta} = 0, \quad 2\varepsilon_{33}^{(k) R} = (B_{3})(B_{3}) - \varepsilon_{3} \varepsilon_{3} = 0, \quad 2\varepsilon_{\alpha 3}^{(k) R} = H^{(k)}_{\alpha} [(B_{\alpha})(B_{3}) - \varepsilon_{\alpha} \varepsilon_{3}] = 0. \] (3.9)

More simple and convenient strain-displacement relationships of the geometrically nonlinear theory of laminate shells can be obtained on the basis if the broken normal hypothesis. To that end, one should ignore the quadratic terms depending on the transverse coordinate \( \alpha_{3} \) in the expressions of (2.6) for the tangential components of the Green–Lagrangian strain tensor. As a result we obtain

\[ 2\varepsilon^{(k\alpha)}_{\alpha \beta} = N^{(k)}_{\alpha}(\alpha_{3}) \left( \frac{1}{H^{(k-1)}_{\alpha}} \varepsilon^{(k-1)}_{\alpha} e_{\beta} + \frac{1}{H^{(k)}_{\alpha}} \varepsilon^{(k)}_{\alpha} e_{\beta} \right) \]

\[ + \frac{1}{H^{(k)}_{\alpha}} \varepsilon^{(k)}_{\alpha} \left( \frac{1}{H^{(k)}_{\beta}} \varepsilon^{(k)}_{\beta} e_{\beta} + \frac{1}{H^{(k)}_{\beta}} \varepsilon^{(k)}_{\beta} e_{\beta} \right) \]

\[ 2\varepsilon^{(k\alpha)}_{\alpha 3} = \beta^{(k)} e_{\alpha} + \frac{1}{H^{(k)}_{\alpha}} \varepsilon^{(k)}_{\alpha} (e_{3} + \beta^{(k)}) + (\alpha_{3} - \delta_{k}) \frac{1}{H^{(k)}_{\alpha}} \varepsilon^{(k\alpha)}_{33}, \quad \varepsilon^{(k\beta)}_{33} = \beta^{(k)} \left( e_{3} + \frac{1}{2} \beta^{(k)} \right). \] (3.10)

These relationships provide exact expressions for arbitrarily large rigid displacements of the shell \( k \)th layer, since \( \varepsilon^{(k) R}_{ij} = 0 \), as was the case for (3.9).

The strain-displacement relationships of the theory of laminate shells constructed on the basis of the broken normal hypothesis satisfy the conditions

\[ \varepsilon^{(k\alpha)}_{\alpha \beta}(\delta_{l}) = \varepsilon^{(k\alpha)}_{\alpha \beta}(\delta_{l}) = \varepsilon^{(k\alpha)}_{\alpha \beta}(\delta_{l}) = \frac{E^{(l)}}{E^{(l)}} \quad (l = k - 1, k), \quad \varepsilon^{(k\alpha)}_{\alpha 3}(\delta_{k}) = \varepsilon^{(k\alpha)}_{\alpha 3}(\delta_{k}) = \varepsilon^{(k\alpha)}_{\alpha 3}(\delta_{k}) = \frac{E^{(k)}}{E^{(k)}}. \] (3.11)
where $E^{(k)}_{\alpha\beta}$ are the tangential stresses on the $k$th layer face surfaces and $\bar{E}^{(k)}_{\alpha3}$ are the transverse shears of the middle surface of the $k$th layer. The geometrical interpretation of the conditions of (3.11) is shown in Fig. 3.

The scalar form of the relationships of (3.10) is given by

$$
\varepsilon^{(k)c}_{\alpha\beta} = N_k(\alpha_3)E^{(k-1)}_{\alpha\beta} + N^*_k(\alpha_3)E^{(k)}_{\alpha\beta}, \quad \varepsilon^{(k)c}_{\alpha3} = N_k(\alpha_3)E^{(k)}_{\alpha3} + N^*_k(\alpha_3)E^{(k)*}_{\alpha3}, \quad \varepsilon^{(k)c}_{33} = E^{(k)}_{33},
$$

(3.12)
\[ F^{(l)}_{\alpha\beta} = \varepsilon^{(l)}_{\alpha\beta} + \eta^{(l)}_{\alpha\beta}, \quad E^{(kl)} = \varepsilon^{(kl)}_{\alpha\beta} + \eta^{(kl)}_{\alpha\beta}, \quad E^{(3)} = \varepsilon^{(3)}_{\alpha\beta} + \eta^{(3)}_{\alpha\beta}. \] (3.13)

\[ e^{(l)}_{\alpha\beta} = \frac{1}{\zeta^{(l)}_{\alpha\beta}}, \quad 2e^{(l)}_{12} = \frac{1}{\zeta^{(l)}_{12}} \lambda^{(l)}_{12} + \frac{1}{\zeta^{(l)}_{21}} \lambda^{(l)}_{21}, \quad e^{(3)}_{33} = \beta^{(3)}_{33}. \]

\[ 2 \varepsilon^{(k)}_{\alpha\beta} = 2 \left( \frac{1}{\zeta^{(k)}_{\alpha\beta}} \right) \beta^{(k)}_{\alpha\beta} = \frac{1}{\zeta^{(k)}_{\alpha\beta}} \beta^{(k)}_{\alpha\beta} - \frac{1}{\zeta^{(k)}_{\alpha\beta}} \beta^{(k)}_{\alpha\beta}, \quad 2 \varepsilon^{(k+)}_{\alpha\beta} = \frac{\eta^{(k+)}_{\alpha\beta}}{\zeta^{(k)}_{\alpha\beta}} \beta^{(k)}_{\alpha\beta} - \frac{1}{\zeta^{(k)}_{\alpha\beta}} \beta^{(k)}_{\alpha\beta}, \]

\[ \eta^{(l)}_{\alpha\beta} = \frac{1}{2(\zeta^{(l)}_{\alpha\beta})^2} \left[ (\lambda^{(l)}_{12})^2 + (\lambda^{(l)}_{21})^2 \right], \quad 2 \eta^{(l)}_{12} = \frac{1}{\zeta^{(l)}_{12}} \left( \lambda^{(l)}_{12} \omega^{(l)}_{12} + \lambda^{(l)}_{21} \omega^{(l)}_{21} + \theta^{(l)}_{12} \phi^{(l)}_{12} \right), \]

\[ 2 \eta^{(k+)}_{\alpha\beta} = \frac{1}{\zeta^{(k)}_{\alpha\beta}} \left( \beta^{(k)}_{\alpha\beta} \lambda^{(k+)}_{\alpha\beta} + \beta^{(k)}_{\alpha\beta} \omega^{(k+)}_{\alpha\beta} - \beta^{(k)}_{\alpha\beta} \theta^{(k)}_{\alpha\beta} \phi^{(k)}_{\alpha\beta} \right), \quad 2 \eta^{(k-)}_{\alpha\beta} = \frac{1}{\zeta^{(k)}_{\alpha\beta}} \left( \beta^{(k)}_{\alpha\beta} \lambda^{(k-)}_{\alpha\beta} + \beta^{(k)}_{\alpha\beta} \omega^{(k-)}_{\alpha\beta} - \beta^{(k)}_{\alpha\beta} \theta^{(k)}_{\alpha\beta} \phi^{(k)}_{\alpha\beta} \right), \]

\[ \eta^{(3)}_{\alpha\beta} = \frac{(\beta^{(3)}_{12})^2 + (\beta^{(3)}_{21})^2 + (\beta^{(3)}_{33})^2}{2}, \quad \lambda^{(l)}_{\alpha\beta} = \frac{1}{A_{\alpha}} \nu^{(l)}_{\alpha\beta} + B_{\gamma} \nu^{(l)}_{\gamma\beta} + k_{\alpha} \nu^{(l)}_{\alpha\beta}, \]

\[ c^{(l)}_{\alpha\beta} = \frac{1}{A_{\alpha}} \nu^{(l)}_{\gamma\beta} + B_{\gamma} \nu^{(l)}_{\gamma\beta} + k_{\alpha} \nu^{(l)}_{\alpha\beta}, \quad \beta^{(l)}_{\alpha\beta} = \frac{1}{A_{\alpha}} \nu^{(k)}_{\alpha\beta} + k_{\alpha} \nu^{(k)}_{\alpha\beta}, \quad \beta^{(k)}_{\alpha\beta} = \frac{u^{(k)}_{\alpha\beta} - u^{(k-1)}_{\alpha\beta}}{h_{k}}, \]

\[ \zeta^{(l)}_{\alpha\beta} = 1 + k_{\alpha} \delta_{l}, \quad \zeta^{(k)}_{\alpha\beta} = 1 + k_{\alpha} \delta_{k}, \quad B_{\alpha} = \frac{1}{A_{1} A_{2}} A_{\gamma\alpha} \quad (l = k - 1, k; \quad \gamma \neq \alpha). \]

4. CONCLUSIONS

In the present paper, the expressions for the Green–Lagrange strain tensor have been constructed in curvilinear coordinates for homogeneous shells on the basis of Timoshenko’s kinematic hypothesis ((2.12) and (2.13)) and for laminate shells on the basis of the broken normal hypothesis ((3.12) and (3.13)). These expressions can be successfully utilized in finite element technology for developing new effective finite elements for homogeneous and laminate shells, since these expressions exactly represent arbitrarily large rigid displacements of the finite elements.

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