Assessment of the sampling surfaces formulation for thermoelectroelastic analysis of layered and functionally graded piezoelectric shells

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ABSTRACT
The article focuses on the implementation of the sampling surfaces (SaS) concept for the three-dimensional (3D) coupled steady-state thermoelectroelastic analysis of layered and functionally graded (FG) piezoelectric shells subjected to thermal loading. The SaS formulation is based on choosing inside the nth layer \( l_n \), not equally spaced SaS parallel to the middle surface, in order to introduce the temperatures, electric potentials, and displacements of these surfaces as basic shell variables. Such choice of unknowns with the consequent use of Lagrange polynomials of degree \( l_n - 1 \) in the assumed distributions of the temperature, electric potential, displacements, and mechanical properties through the thickness of the layer leads to the robust FG piezoelectric shell formulation. The SaS are located inside each layer at Chebyshev polynomial nodes, which permits one to minimize uniformly the error due to the Lagrange interpolation. As a result, the SaS formulation can be applied efficiently to deriving the analytical solutions for FG piezoelectric shells, which asymptotically approach the 3D exact solutions of thermoelectroelasticity as the number of SaS tends to infinity.

1. Introduction
The functionally graded (FG) piezoelectric materials are widely used in engineering due to their advantages compared to traditional piezoelectric materials (see, e.g., [1, 2]) and, therefore, the analysis of FG piezoelectric structures is of practical interest. However, the three-dimensional (3D) analysis of FG plates and shells is not a simple task because the material properties depend on the transverse coordinate and some specific assumptions concerning their variations in the thickness direction are required [3, 4]. As a result, the Pagano approach [5] and the state space approach [6, 7] cannot be applied to 3D exact solutions for FG piezoelectric structures directly. In practice, the shell is artificially divided into a large number of the individual layers with constant material properties through their thicknesses [8–11] following a technique proposed by Solidatos and Hadjiigeorgiou [12] (the details can be found in review [4]). One can see that the solutions derived following such a technique are not exact; they are approximate. At the same time, the obtaining of 3D exact solutions for FG piezoelectric plates and shells is still possible. To solve this problem, one can assume that the material properties are distributed through the thickness of the plate according to the exponential law [13] employing the state space approach or the power law [14] by using the Frobenius method [15]. It is apparent that the latter technique is not easy to carry out. The method of asymptotic homogenization [16] also can be applied for the thermoelectroelastic analysis of smart periodic composite structures [17, 18].

In the literature, there is a powerful tool to readily overcome the above-mentioned difficulties. This is a sampling surfaces (SaS) concept developed recently by the authors. The SaS concept was first utilized for the 3D elasticity analysis of laminated composite plates and shells [19, 20]. Then, the SaS formulation was extended to the heat conduction theory [21], thermoelasticity [22], electroelasticity [23], and thermoelectroelasticity [24, 25] as well. According to the SaS concept, one chooses the arbitrarily located surfaces inside the nth layer parallel to the middle surface \( \Omega_{nn}^{(1)}, \Omega_{nn}^{(2)}, \ldots, \Omega_{nn}^{(l_n)} \) in order to introduce the temperatures \( T^{(1)}, T^{(2)}, \ldots, T^{(l_n)} \), electric potentials \( \varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(l_n)} \), and displacement vectors \( u^{(1)}, u^{(2)}, \ldots, u^{(l_n)} \) of these surfaces as basic shell unknowns, where \( l_n \) is the number of SaS of the nth layer \( (I_n \geq 3) \). Such choice of shell unknowns with the consequent use of the Lagrange polynomials of degree \( l_n - 1 \) in through-thickness distributions of the temperature, electric potential, and displacements for each layer leads to a very compact form of the governing equations of the SaS layered shell formulation. The implementation of the SaS concept for FG plates and shells [26–28] requires additionally the use of the same Lagrangian through-thickness interpolations of material coefficients inside each layer. Note that the SaS formulation has been implemented only for the thermal stress analysis of FG piezoelectric plates [28]. The present article is intended to extend the SaS formulation to the FG layered thermopiezoelectric shells.

It should be noticed that the SaS shell formulation with equally spaced SaS does not work properly with the Lagrange
polynomials of high degree because of Runge’s phenomenon [29]. This phenomenon yields the wild oscillation at the edges of the interval when the user deals with some specific functions similar to the shell metric functions. If the number of equispaced nodes is increased then the oscillations become even larger. However, the use of the Chebyshev polynomial nodes [30] inside the shell body can help to improve significantly the behavior of the Lagrange polynomials of high degree because such a choice permits one to minimize uniformly the error due to the Lagrange interpolation. This fact gives an opportunity to obtain the displacements and stresses with a prescribed accuracy employing the sufficiently large number of SaS. It means, in turn, that the solutions based on the SaS concept asymptotically approach the 3D exact solutions of thermoelastoelectricity as the number of SaS $I_n \to \infty$.

The origins of the SaS concept can be found in contributions [31, 32] in which three, four, and five equally spaced SaS are employed. The SaS formulation with the arbitrary number of equispaced SaS is considered in [33]. The more general approach to the SaS formulation with the arbitrary number of SaS defined as:

$$\theta_{3}^{(n)} = \theta_{3}^{(n-1)}, \quad \theta_{3}^{(n)l_0} = \theta_{3}^{[n]} ,$$

$$\theta_{3}^{(n)mn} = \frac{1}{2} \left( \theta_{3}^{[n-1]} + \theta_{3}^{[n]} \right) - \frac{1}{2} h^{(n)} \cos \left( \frac{\pi \cdot 2m - 3}{2(N - 2)} \right) ,$$

where $\theta_{3}^{(n)}$ and $\theta_{3}^{[n]}$ are the transverse coordinates of layer interfaces $\Omega^{[n-1]}$ and $\Omega^{[n]}$ depicted in Figure 1; $h^{(n)} = \theta_{3}^{[n]} - \theta_{3}^{[n-1]}$ is the thickness of the nth layer. It is worth noting that the transverse coordinates of inner SaS $\theta_{3}^{(n)mn}$ coincide with coordinates of the Chebyshev polynomial nodes [30]. This fact has a great meaning for a convergence of the SaS method [19, 20].

2. Description of displacement and strain fields

Consider a layered shell of the thickness $h$. Let the middle surface $\Omega$ be described by orthogonal curvilinear coordinates $\theta_1$ and $\theta_2$, which are referred to the lines of principal curvatures of its surface. The coordinate $\theta_3$ is oriented along the unit vector $\mathbf{e}_3(\theta_1, \theta_2)$ normal to the middle surface. Introducing the following notations: $\mathbf{e}_3(\theta_1, \theta_2)$ are the orthonormal base vectors of the middle surface; $A_\alpha(\theta_1, \theta_2)$ are the coefficients of the first fundamental form; $\kappa_\alpha(\theta_1, \theta_2)$ are the principal curvatures of the middle surface; $c_\alpha = 1 + \kappa_\alpha \theta_3$ are the components of the shifter tensor; $c_{\alpha}^{(n)i}$ $(\theta_1, \theta_2)$ are the components of the shifter tensor at SaS defined as:

$$c_{\alpha}^{(n)i} = c_\alpha \left( \theta_{3}^{(n)l_0} \right) = 1 + \kappa_\alpha \theta_{3}^{(n)l_0},$$

where $\theta_{3}^{(n)l_0}$ are the transverse coordinates of SaS of the $n$th layer $\Omega^{(n)}$ given by:

$$\theta_{3}^{(n)} = \theta_{3}^{[n-1]}, \quad \theta_{3}^{(n)l_0} = \theta_{3}^{[n]},$$

$$\theta_{3}^{(n)mn} = \frac{1}{2} \left( \theta_{3}^{[n-1]} + \theta_{3}^{[n]} \right) - \frac{1}{2} h^{(n)} \cos \left( \frac{\pi \cdot 2m - 3}{2(N - 2)} \right) ,$$

where $\theta_{3}^{(n-1)}$ and $\theta_{3}^{[n]}$ are the transverse coordinates of layer interfaces $\Omega^{[n-1]}$ and $\Omega^{[n]}$ depicted in Figure 1; $h^{(n)} = \theta_{3}^{[n]} - \theta_{3}^{[n-1]}$ is the thickness of the $n$th layer. It is worth noting that the transverse coordinates of inner SaS $\theta_{3}^{(n)mn}$ coincide with coordinates of the Chebyshev polynomial nodes [30]. This fact has a great meaning for a convergence of the SaS method [19, 20].

Figure 1. Geometry of the laminated shell.

Here and in the following developments, the index $n$ identifies the belonging of any quantity to the $n$th layer and runs from 1 to $N$, where $N$ is the number of layers; the index $m_n$ identifies the belonging of any quantity to the inner SaS of the $n$th layer and runs from 2 to $I_n - 1$, where $I_n$ is the number of SaS of the $n$th layer; the indices $i, j, k, l, m$ describe all SaS of the $n$th layer and run from 1 to $I_n$; Latin tensorial indices $i$, $j$, $k$, $l$ range from 1 to 3; Greek indices $\alpha$, $\beta$ range from 1 to 2.

The relation between the displacement vector $u$ and the strain tensor $\varepsilon$ is given by:

$$\varepsilon = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right] ,$$

where $\nabla$ is the nabla operator.

In the orthonormal basis $\mathbf{e}_i$, the strain-displacement equations (3) can be written as:

$$2\varepsilon_{\alpha\beta} = \frac{1}{\epsilon_\alpha} \lambda_{\alpha\beta} + \frac{1}{\epsilon_\alpha} \lambda_{\beta\alpha},$$

$$2\varepsilon_{\alpha3} = \frac{1}{\epsilon_\alpha} \lambda_{3\alpha} + u_{\alpha3}, \quad \varepsilon_{33} = u_{33},$$

where $u_i$ and $\varepsilon_{ij}$ are the displacement and strain components; $\lambda_{\alpha\beta}$ are the strain parameters [22] expressed in terms of displacements as:

$$\lambda_{\alpha\alpha} = \frac{1}{A_\alpha} u_{\alpha\alpha} + B_{\alpha} u_{\beta\beta} + k_{\alpha} u_{33} \text{ for } \beta \neq \alpha ,$$

$$\lambda_{\beta\alpha} = \frac{1}{A_\alpha} u_{\beta\alpha} - B_{\alpha} u_{\beta\beta} \text{ for } \beta \neq \alpha ,$$

$$\lambda_{3\alpha} = \frac{1}{A_\alpha} u_{3\alpha} - k_{\alpha} u_{33}, \quad B_{\alpha} = \frac{1}{A_\alpha} A_\beta A_{\alpha\beta} \text{ for } \beta \neq \alpha ,$$

where the symbol $(\ldots)_{,\lambda}$ stands for the partial derivatives with respect to coordinates $\theta_\lambda$.

We start now with the first two assumptions of the proposed layered thermopiezoelectric shell formulation. Let us assume that the displacement and strain fields are distributed through the thickness of the nth layer as follows:

$$u_i^{(n)} = \sum_{i_n} L_i^{(n)i} u_{i}^{(n)i}, \quad \theta_3^{[n-1]} \leq \theta_3^{(n)} \leq \theta_3^{[n]},$$

$$\varepsilon_{ij}^{(n)} = \sum_{i_n} L_i^{(n)ij} \varepsilon_{ij}^{(n)i}, \quad \theta_3^{[n-1]} \leq \theta_3^{(n)} \leq \theta_3^{[n]},$$

where $L_i^{(n)i}$ and $L_i^{(n)ij}$ are the coefficients of the first and second kind of the Lagrange polynomials, respectively.
where \( u^{(n)i}_i \) and \( \epsilon^{(ni)}_i \) are the displacements and strains of SaS of the \( n \)th layer \( \Omega^{(n)} \), \( L^{(n)i} \) are the Lagrange polynomials of degree \( l_n \) given by:

\[
\begin{align*}
\epsilon^{(ni)}_i &= e_i \left( \theta_3^{(ni)} \right), \\
L^{(n)i} &= \prod_{\beta \neq i} \frac{\theta_3^{(n)i} - \theta_3^{(n)\beta}}{\theta_3^{(n)i} - \theta_3^{(n)\beta}}.
\end{align*}
\]

(8) (9) (10)

The use of Eqs. (4)–(6) and (8)–(10) leads to the following relations for strains of SaS of the \( n \)th layer:

\[
\begin{align*}
2\epsilon^{(n)i}_a &= \frac{1}{\epsilon^{(n)i}_a} \epsilon^{(n)i}_a - \beta^{(n)i}_a, \\
2\epsilon^{(n)i}_a &= \frac{1}{\epsilon^{(n)i}_a} \epsilon^{(n)i}_a + \beta^{(n)i}_a, \\
\epsilon^{(n)i}_a &= \beta^{(n)i}_a,
\end{align*}
\]

(11)

where \( \beta^{(n)i}_a \) are the strain parameters of SaS of the \( n \)th layer; \( \beta^{(n)i}_a \) are the values of the derivative of displacements with respect to thickness coordinate \( \theta_3 \) at SaS defined as:

\[
\begin{align*}
\lambda^{(n)i}_a &= \lambda^{(n)i}_a \left( \theta_3^{(ni)} \right) = \frac{1}{A^a} \frac{\partial M^{(ni)i}}{\partial \theta_3^{(ni)}} + B^a a^{(ni)i}, \\
+ k_a u^{(ni)i}_i & \text{ for } \beta \neq \alpha, \\
\lambda^{(n)i}_a &= \lambda^{(n)i}_a \left( \theta_3^{(ni)} \right) = \frac{1}{A^a} \frac{\partial M^{(ni)i}}{\partial \theta_3^{(ni)}} - B^a a^{(ni)i} & \text{ for } \beta \neq \alpha, \\
\lambda^{(n)i}_a &= \lambda^{(n)i}_a \left( \theta_3^{(ni)} \right) = \frac{1}{A^a} \frac{\partial M^{(ni)i}}{\partial \theta_3^{(ni)}} - k_a u^{(ni)i}_i, \\
\rho^{(n)i}_a &= \rho^{(n)i}_a \left( \theta_3^{(ni)} \right) = \sum_{j_n} M^{(ni)j_n} \left( \theta_3^{(ni)} \right) u^{(ni)j_n}_i, \\
\end{align*}
\]

(12) (13)

where \( M^{(ni)}_j = L^{(ni)}_j \) are the derivatives of the Lagrange polynomials, which are calculated at SaS as follows:

\[
\begin{align*}
M^{(ni)j_n} &= \left( \theta_3^{(ni)} \right) = \frac{1}{\theta_3^{(ni)}} \prod_{\beta \neq i} \frac{\theta_3^{(ni)} - \theta_3^{(n)\beta}}{\theta_3^{(ni)} - \theta_3^{(n)\beta}} \\
M^{(ni)j_n} &= - \sum_{j_n \neq i_n} M^{(ni)j_n} \left( \theta_3^{(ni)} \right).
\end{align*}
\]

(14)

It is seen from Eq. (13) that the key functions \( \rho^{(n)i}_a \) of the layered shell formulation are represented as a linear combination of displacements of SaS of the \( n \)th layer \( u^{(n)i}_i \).

Remark: Strain-displacement relationships (11)–(13) exactly represent rigid-body motions of a layered shell in any convected curvilinear coordinate system. This statement can be proved following [20].

3. Description of electric field

The relation between the electric potential \( \varphi \) and the electric field \( E \) is written as:

\[
E = -\nabla \varphi.
\]

(15)

In the orthonormal basis \( e_i \), it is expressed as:

\[
E_a = -\frac{1}{A_a e_a} \varphi_a, \quad E_3 = -\varphi_3.
\]

(16)

Now, we introduce the third and fourth assumptions of the proposed layered thermopiezoelectric shell formulation. Let us assume that the electric potential and the electric field are distributed through the thickness of the \( n \)th layer as follows:

\[
\varphi^{(n)} = \sum_{i_n} L^{(ni)i} \varphi^{(ni)i}, \quad \theta_3^{(n-1)} \leq \theta_3 \leq \theta_3^{[n]},
\]

(17)

\[
E^{(n)i} = \sum_{i_n} L^{(ni)i} E^{(ni)i}, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]},
\]

(18)

where \( \varphi^{(ni)i} \) are the potential of SaS of the \( n \)th layer; \( E^{(ni)i} \) are the components of the electric field at the same SaS given by:

\[
\varphi^{(ni)i} = \varphi \left( \theta_3^{(ni)} \right), \\
E^{(ni)i} = E_i \left( \theta_3^{(ni)} \right).
\]

(19) (20)

Using Eqs. (16), (17), (19), and (20) one obtains the following relations:

\[
E^{(ni)i}_a = -\frac{1}{A_a e_a} \varphi^{(ni)i}_a, \\
E^{(ni)i}_3 = -\sum_{j_n} M^{(ni)j_n} \left( \theta_3^{(ni)} \right) \varphi^{(ni)j_n}_i.
\]

(21) (22)

As can be seen, Eq. (22) is similar to Eq. (13). Thus, the normal components of the electric field on SaS of the \( n \)th layer \( E^{(ni)i}_3 \) are represented as a linear combination of electric potentials of SaS of the same layer \( \varphi^{(ni)i} \).

4. Description of temperature field

The relation between the temperature \( T \) and the temperature gradient \( \Gamma \) is given by:

\[
\Gamma = \nabla T.
\]

(23)

In a component form, it can be written as:

\[
\Gamma_a = \frac{1}{A_a e_a} T_a, \quad \Gamma_3 = T_3.
\]

(24)

The following step consists of a choice of the suitable approximation of the temperature and temperature gradient through the thickness of the \( n \)th layer. It is apparent that the temperature and temperature gradient distributions should be chosen similar to electric potential and electric field distributions (17) and (18). Therefore, the next two assumptions of the proposed layered thermopiezoelectric shell formulation are:

\[
T^{(n)i} = \sum_{i_n} L^{(ni)i} T^{(ni)i}, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]},
\]

(25)

\[
\Gamma^{(n)i}_i = \sum_{i_n} L^{(ni)i} \Gamma^{(ni)i}_i, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]},
\]

(26)

where \( T^{(ni)i} \) are the temperatures of SaS of the \( n \)th layer; \( \Gamma^{(ni)i}_i \) are the components of the temperature gradient at
the same SαS defined as:

\[ T^{(n)i \alpha} = T \left( \theta_3^{(n)i \alpha} \right), \]  
\[ \Gamma_i^{(n)i \alpha} = \Gamma_i \left( \theta_3^{(n)i \alpha} \right). \]

The use of Eqs. (24), (25), (27), and (28) yields:

\[ \Gamma_i^{(n)i \alpha} = \frac{1}{A_{n \alpha}^{(n)i \alpha}} T^{(n)i \alpha}, \]
\[ \Gamma_3^{(n)i \alpha} = \sum_{kn} M^{(n)kj} \left( \theta_3^{(n)kj} \right) T^{(n)j \alpha}. \]

It is seen from Eq. (30) that the normal components of the convective heat transfer coefficient \( q \) are presented as a linear combination of temperatures of SαS of the same layer \( T^{(n)i \alpha} \).

5. Variational formulation of heat conduction problem

The variational equation for the thermal layered shell is written as:

\[ \delta I = 0, \]

where \( I \) is the basic functional of the heat conduction theory given by:

\[ I = \frac{1}{2} \int_{\Omega} \sum_{n} \int_{\Omega} \left( \mathbf{q}_i \Gamma_i^{(n)i \alpha} \right) A_1 A_2 \mathbf{c}_1 \mathbf{c}_2 dT \theta_2 d\theta_3 \]
\[ - \int_{\Omega} \left[ \mathbf{q}_n T + \frac{1}{2} \mu (T - \hat{T})^2 \right] d\Omega. \]

where \( \mathbf{q}_i^{(n)} \) are the components of the heat flux vector of the nth layer; \( \mathbf{q}_n \) is the specified heat flux on the part of the boundary surface \( \Omega; \mu \) is the convective heat transfer coefficient; and \( \hat{T} \) is the reference temperature for the convective transfer. Here and in the following derivations, the summation on repeated Latin indices is implied.

Substituting Eq. (26) in Eq. (32) and introducing the heat flux resultants:

\[ Q_i^{(n)i \alpha} = \int_{\Omega} \mathbf{q}_i^{(n)j} L^{(n)i \alpha} \mathbf{c}_1 \mathbf{c}_2 d\theta_3, \]

one obtains:

\[ I = \frac{1}{2} \int_{\Omega} \sum_{n} \sum_{\alpha_k} Q_i^{(n)i \alpha} \Gamma_i^{(n)i \alpha} A_1 A_2 \theta_1 d\theta_2 d\theta_3 \]
\[ - \int_{\Omega} \left[ \mathbf{q}_n T + \frac{1}{2} \mu (T - \hat{T})^2 \right] d\Omega. \]

As constitutive equations, we accept the Fourier heat conduction equations:

\[ q_i^{(n)} = -k_i^{(n)} \Gamma_j, \quad \theta_3^{(n-1)} \leq \theta_3 \leq \theta_3^{(n)} \]

where \( k_i^{(n)} \) are the thermal conductivities of the nth layer.

Introducing the seventh assumption of the thermal FG layered shell formulation, let us assume that the thermal conductivity coefficients are distributed through the thickness of the nth layer according to the following law:

\[ \theta_3^{(n)} = \sum_k \frac{L^{(n)k \alpha}}{\theta_3^{(n-1)}}, \quad \theta_3^{(n)} \leq \theta_3 \leq \theta_3^{(n)}, \]

which is extensively utilized in this article, where \( k_i^{(n)} \) are the values of the thermal conductivity tensor on SαS of the nth layer.

The use of constitutive equations (35) and through-thickness distributions (26) and (36) in Eq. (33) results in:

\[ Q_i^{(n)i \alpha} = - \sum_{kn} \Lambda_i^{(n)ij \alpha} k_i^{(n)} \Gamma_j \]

where

\[ \Lambda_i^{(n)ij \alpha} = \int_{\Omega} \left[ C_i^{(n)} + \frac{1}{2} \mu (T - \hat{T})^2 \right] d\Omega. \]

6. Variational formulation of thermopiezoelectric shell problem

The variational equation for the layered thermopiezoelectric shell in the case of conservative loading can be written as:

\[ \delta \Pi = 0, \]

where \( \Pi \) is the basic functional of thermopiezoelectricity given by:

\[ \Pi = \int_{\Omega} \sum_{n} \int_{\Omega} \Phi_i^{(n)} A_1 A_2 \mathbf{c}_1 \mathbf{c}_2 dT \theta_2 d\theta_3 \]
\[ - W, \]

\[ \Phi_i^{(n)} = \frac{1}{2} \left( \sigma_i^{(n)E_j} - D_i^{(n)} \epsilon_j^{(n)} - \eta_i^{(n)} \phi_j^{(n)} \right), \]

\[ W = \int_{\Omega} \left[ \mathbf{c}_1 \mathbf{c}_2 \left( p_i^+ u_i^+ - Q^- \phi^- \right) - \mathbf{c}_1 \mathbf{c}_2 \left( p_i^- u_i^- + Q^+ \phi^+ \right) \right] \]
\[ \times A_1 A_2 \theta_1 d\theta_2 + W_3, \]

where \( \Phi_i^{(n)} \) is the enthalpy function [37] of the nth layer; \( \sigma_i^{(n)} \) are the stresses of the nth layer; \( D_i^{(n)} \) are the components of the dielectric displacement vector of the nth layer; \( \eta_i^{(n)} \) is the entropy density of the nth layer; \( u_i^- = u_i^{(1)} \) and \( u_i^+ = u_i^{(N)} \) are the displacements of outer surfaces; \( \phi^- = \phi^{(1)} \) and \( \phi^+ = \phi^{(N)} \) are the electric potentials of outer surfaces; \( C_i^- = C_i^{(1)} \) and \( C_i^+ = C_i^{(N)} \) are the components of the shifter tensor on outer surfaces; \( p_i^+ \) and \( p_i^- \) are the external mechanical loads acting on outer surfaces; \( Q^- \) and \( Q^+ \) are the specified electric charges on outer surfaces; \( W_3 \) is the work done by thermal and electromechanical loads applied to the edge boundary surface \( \Sigma; \Theta_i^{(n)} \) is the temperature rise of the nth layer from the initial reference temperature \( T_0 \) that can be presented according to Eq. (25) as follows:

\[ \Theta_i^{(n)} = \sum_k L^{(n)k \alpha} \Theta_i^{(n)k \alpha}, \quad \theta_3^{(n-1)} \leq \theta_3 \leq \theta_3^{(n)} \]

where \( \Theta_i^{(n)k \alpha} = T^{(n)k \alpha} - T_0 \) is the temperature rise of the SαS of the nth layer.
Substituting the through-thickness distributions (7), (18), and (43) in Eqs. (40) and (41), and introducing stress resultants:

\[ H_{ij}^{(n)i} = \int_{\gamma^3}^{\gamma^1} \sigma_{ij}^{(n)i} L^{(n)i} c_1 c_2 d\theta_3, \]  
(44)

electric displacement resultants:

\[ R_i^{(n)i} = \int_{\gamma^3}^{\gamma^1} D_i^{(n)i} L^{(n)i} c_1 c_2 d\theta_3 \]  
(45)

and entropy resultants:

\[ S^{(n)i} = \int_{\gamma^3}^{\gamma^1} \eta^{(n)} L^{(n)i} c_1 c_2 d\theta_3, \]  
(46)

one derives:

\[ \Pi = \frac{1}{2} \int_\Omega \sum_{\gamma^3}^{\gamma^1} \left( H_{ij}^{(n)i} \sigma_{ij}^{(n)i} - R_i^{(n)i} \gamma_{ij}^{(n)} - S^{(n)i} \eta^{(n)} \right) \times A_1 A_2 d\theta_1 d\theta_2 - W. \]  
(47)

For simplicity, we consider the case of linear piezoelectric materials. Therefore, the constitutive equations \([37]\) are expressed as:

\[ \sigma_{ij}^{(n)} = C_{ijkl}^{(n)} \varepsilon_{kl}^{(n)} - \varepsilon_{ij}^{(n)} E_k^{(n)} - \gamma_{ij}^{(n)} \Theta_1^{(n)}, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]}, \]  
(48)

\[ D_i^{(n)} = \varepsilon_{ik}^{(n)} E_k^{(n)} + r_i^{(n)} \Theta_1^{(n)}, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]}, \]  
(49)

\[ \eta^{(n)} = \gamma_{ik}^{(n)} E_k^{(n)} + \chi^{(n)} \Theta_1^{(n)}, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]}, \]  
(50)

where \( C_{ijkl}^{(n)} \) are the elastic constants of the \( n \)th layer; \( \varepsilon_{ij}^{(n)} \) are the piezoelectric constants of the \( n \)th layer; \( \gamma_{ij}^{(n)} \) are the thermal stress coefficients of the \( n \)th layer; \( \varepsilon_{ik}^{(n)} \) are the dielectric constants of the \( n \)th layer; \( r_i^{(n)} \) are the pyroelectric constants of the \( n \)th layer; and \( \chi^{(n)} \) is the entropy-temperature coefficient of the \( n \)th layer defined as:

\[ \chi^{(n)} = \rho^{(n)} c_v^{(n)} / T_0, \]  
(51)

where \( \rho^{(n)} \) and \( c_v^{(n)} \) are the mass density and the specific heat per unit mass of the \( n \)th layer at constant strain.

Finally, we introduce the last assumptions of the SaS FG thermopiezoelectric shell formulation. Let the material constants be distributed through the thickness of the \( n \)th layer as accepted throughout this article:

\[ C_{ijkl}^{(n)} = \sum_{\gamma^3}^{\gamma^1} L^{(n)i} C_{ijkl}^{(n)i} c_1 c_2 d\theta_3, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]}, \]  
(52)

\[ \varepsilon_{ij}^{(n)} = \sum_{\gamma^3}^{\gamma^1} L^{(n)i} \varepsilon_{ij}^{(n)i} c_1 c_2 d\theta_3, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]}, \]  
(53)

\[ \gamma_{ij}^{(n)} = \sum_{\gamma^3}^{\gamma^1} L^{(n)i} \gamma_{ij}^{(n)i} c_1 c_2 d\theta_3, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]}, \]  
(54)

\[ \varepsilon_{ik}^{(n)} = \sum_{\gamma^3}^{\gamma^1} L^{(n)i} \varepsilon_{ik}^{(n)i} c_1 c_2 d\theta_3, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]}, \]  
(55)

\[ r_i^{(n)} = \sum_{\gamma^3}^{\gamma^1} L^{(n)i} r_i^{(n)i} c_1 c_2 d\theta_3, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]}, \]  
(56)

\[ \chi^{(n)} = \sum_{\gamma^3}^{\gamma^1} L^{(n)i} \chi^{(n)i}, \quad \theta_3^{[n-1]} \leq \theta_3 \leq \theta_3^{[n]}, \]  
(57)

where \( C_{ijkl}^{(n)i} \), \( \varepsilon_{ij}^{(n)i} \), \( \gamma_{ij}^{(n)i} \), \( \varepsilon_{ik}^{(n)i} \), \( r_i^{(n)i} \), \( \chi^{(n)i} \) are the values of material constants on SaS of the \( n \)th layer.

The use of constitutive equations (48)–(50) and through-thickness distributions (7), (18), (43), and (52)–(57) in Eqs. (44)–(46) yields:

\[ H_{ij}^{(n)i} = \sum_{\gamma^3}^{\gamma^1} A_{ij}^{(n)i} \left( C_{ijkl}^{(n)} \varepsilon_{kl}^{(n)i} - \varepsilon_{ij}^{(n)i} E_k^{(n)i} - \gamma_{ij}^{(n)i} \Theta_1^{(n)i} \right), \]  
(58)

\[ R_i^{(n)i} = \sum_{\gamma^3}^{\gamma^1} A_{ik}^{(n)i} \left( \varepsilon_{ik}^{(n)i} E_k^{(n)i} + r_i^{(n)i} \Theta_1^{(n)i} \right), \]  
(59)

\[ S^{(n)i} = \sum_{\gamma^3}^{\gamma^1} A_{ik}^{(n)i} \left( \gamma_{ik}^{(n)i} E_k^{(n)i} + \chi^{(n)i} \Theta_1^{(n)i} \right). \]  
(60)

The definite integrals \( A_{ij}^{(n)i} \), \( A_{ik}^{(n)i} \), \( A_{ik}^{(n)i} \) are defined by Eq. (38).

7. Analytical solution for axisymmetric FG anisotropic cylindrical shells

In this section, we study a FG layered cylindrical shell subjected to axisymmetric thermal and electromechanical loads. The boundary conditions for the simply supported shell with electrically grounded edges maintained at the reference temperature are written as:

\[ \Theta = \varphi = \sigma_{11} = \sigma_{12} = u_3 = 0 \text{ at } \theta_1 = 0 \text{ and } \theta_1 = L, \]  
(61)

where \( \theta_1 \) is the axial coordinate; \( L \) is the length of the shell. To satisfy the boundary conditions, we search the analytical solution by a method of the Fourier series expansion

\[ \Theta = \sum_r \theta_1 \cos \frac{r\pi \theta_1}{L}, \]  
(62)

\[ \varphi = \sum_r \varphi_1 \sin \frac{r\pi \theta_1}{L}, \]  
(63)

\[ u_1 = \sum_r u_{1r} \cos \frac{r\pi \theta_1}{L}, \quad u_2 = \sum_r u_{2r} \cos \frac{r\pi \theta_1}{L}, \]  
(64)

where \( r \) is the wave number. The external electromechanical loads are also expanded in Fourier series.
Substituting the Fourier series (62) in Eq. (34) and using Eqs. (29), (30), (37), (42), and taking into consideration the identity:

$$\sum_{i_n} M^{(n)i_n}(\theta_j) = 0, \quad (65)$$

one obtains:

$$J = \sum_{r} f_r(\Theta_r^{(n)i_n}). \quad (66)$$

Invoking the variational equation (13), we arrive at the system of linear algebraic equations:

$$\frac{\partial J}{\partial \Theta_r^{(n)i_n}} = 0, \quad (67)$$

of order $K$, where $K = \sum_n I_n + N + 1$. Therefore, the temperature rises $\Theta_r^{(n)i_n}$ of SaS of the $r$th layer can be found by using a method of Gaussian elimination.

Next, substituting Fourier series (63), (64) and Fourier series related to electromechanical loadings in Eqs. (42), (47) and taking into account Eqs. (11)–(13), (21), (22), (58)–(60), we have:

$$\Pi = \sum_{r} \Pi_r(\Theta_r^{(n)i_n}, \varphi_r^{(n)i_n}, u_r^{(n)i_n}). \quad (68)$$

The use of the variational equation (39) leads to the following system of linear algebraic equations in terms of electric potentials $\varphi_r^{(n)i_n}$ and displacements $u_r^{(n)i_n}$ of SaS of the $r$th layer:

$$\frac{\partial \Pi_r}{\partial \varphi_r^{(n)i_n}} = 0, \quad (69)$$

$$\frac{\partial \Pi_r}{\partial u_r^{(n)i_n}} = 0, \quad (70)$$

of order $4K$. The linear system (69) and (70) is solved again through the method of Gaussian elimination.

The described algorithm was performed with the Symbolic Math Toolbox, which incorporates symbolic computations into the numeric environment of MATLAB. This permits the obtaining of analytical solutions for axisymmetric FG layered anisotropic cylindrical shells in the framework of the SaS thermoelastoelectric shell formulation, which asymptotically approach the 3D exact solutions of thermopiezoelectricity as the number of SaS goes to infinity.

### 7.1. Angle-ply cylindrical shell covered with FG piezoelectric layers

Consider a two-layer angle-ply cylindrical shell [45/−45] composed of the graphite/epoxy composite and covered with FG piezoelectric layers on its bottom and top surfaces. This means that the hybrid four-layer cylindrical shell [PZT/45/−45/PZT] with ply thicknesses $[0.25h/0.25h/0.25h/0.25h]$ is considered. The material properties of the graphite-epoxy composite [25] are:

- $E_L = 172.5$ GPa, $E_T = 6.9$ GPa, $G_{LT} = 3.45$ GPa,
- $G_{TT} = 1.38$ GPa, $v_{LT} = v_{TT} = 0.25$,
- $\alpha_L = 0.57 \times 10^{-6}$ 1/K, $\alpha_T = 35.6 \times 10^{-6}$ 1/K,
- $\varepsilon_1 = 3.095 \times 10^{-11}$ F/m, $\varepsilon_T = 2.653 \times 10^{-11}$ F/m,
- $k_L = 36.42$ W/mK, $k_T = 0.96$ W/mK, $\rho = 1800$ Kg/m$^3$,
- $c_v = 900$ J/KgK,

where L and T stand for the fiber and transverse directions. Concerning both FG piezoelectric layers it is assumed that their material properties are distributed through the shell thickness according to the power law, that is,

$$\varepsilon^{(i)} = \varepsilon^{(0)}(3 + 4z)^i, \quad -0.5 \leq z \leq -0.25,$$

$$\varepsilon^{(4)} = \varepsilon^{(0)}(3 - 4z)^4, \quad 0.25 \leq z \leq 0.5,$$

$$\varepsilon^{(m)} = \left[ c_{ijkl}^{(m)} , c_{ijkl}^{(m)} , f_{ij}^{(m)} , f_{ij}^{(m)} , r_i^{(m)} , r_i^{(m)} , k_{ij}^{(m)} , k_{ij}^{(m)} , \rho^{(m)} , \rho^{(m)} , c_v^{(m)} \right], \quad (71)$$

where $m = 0, 1$ and 4; $c_{ijkl}^{(0)}, c_{ijkl}^{(0)}, f_{ij}^{(0)}, f_{ij}^{(0)}, r_i^{(0)}, r_i^{(0)}, k_{ij}^{(0)}, k_{ij}^{(0)}, \rho^{(0)}, \rho^{(0)}$, and $c_v^{(0)}$ are the material properties of the PZT-5A given in [25] and Table 1; $\alpha$ is the material gradient index; $z = \theta_j / h$ is the dimensionless thickness coordinate.

The shell is loaded on the top surface by sinusoidally distributed temperature loading. The boundary conditions on the bottom and top surfaces are considered as follows:

$$\Theta^+ = \Theta_0 \sin \frac{\pi \theta_1}{L}, \quad D^+ = \sigma_0^+ = \sigma_3^+ = \sigma_3^+ = 0,$$

$$\Theta^- = D^- = \sigma_1^- = \sigma_2^- = \sigma_3^- = 0, \quad (72)$$

where $\Theta_0 = 1$ K and $T_0 = 293$ K. The geometric parameters of the shell are taken to be $L = 4$ m and $R = 1$ m, where $R$ is the radius of the middle surface. To analyze the derived results efficiently, we introduce dimensionless variables as functions of the
Table 2. Results for a FG four-layer anisotropic cylindrical shell with $R/h = 2$ and $\alpha = 1$.

<table>
<thead>
<tr>
<th>$l_n$</th>
<th>$\tilde{\Theta}(0.25)$</th>
<th>$\tilde{\psi}(0.375)$</th>
<th>$\tilde{\psi}(0.5)$</th>
<th>$\tilde{\psi}(2.0)$</th>
<th>$\tilde{u}_n(0.5)$</th>
<th>$\tilde{u}_2(0.5)$</th>
<th>$\tilde{\sigma}(0.5)$</th>
<th>$\tilde{\sigma}(1.25)$</th>
<th>$\tilde{\alpha}(0.25)$</th>
<th>$\tilde{\alpha}(0.375)$</th>
<th>$\tilde{\alpha}(1.25)$</th>
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Table 3. Results for a FG four-layer anisotropic cylindrical shell with $R/h = 10$ and $\alpha = 1$.

<table>
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<th>$l_n$</th>
<th>$\tilde{\Theta}(0.25)$</th>
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<th>$\tilde{u}_n(0.5)$</th>
<th>$\tilde{u}_2(0.5)$</th>
<th>$\tilde{\sigma}(0.5)$</th>
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<td>-1.90133</td>
<td>-4.50511</td>
<td>-0.93028</td>
</tr>
</tbody>
</table>

z-coordinate defined as:

$$\tilde{\Theta} = \Theta(L/2, z)/\Theta_0, \quad \tilde{\psi} = 10^3\tilde{d}\psi(L/2, z)/h\alpha\Theta_0,$$

$$\tilde{D}_3 = 10^3\tilde{S}D_3(L/2, z)/E_r\alpha\Theta_0,$$

$$\tilde{u}_a = 100u_a(0, z)/R\alpha\Theta_0, \quad \tilde{u}_3 = 10u_3(L/2, z)/R\alpha\Theta_0,$$

$$\tilde{\sigma}_{11} = \sigma_{11}(L/2, z)/E_r\alpha\Theta_0, \quad \tilde{\sigma}_{22} = \sigma_{22}(L/2, z)/E_r\alpha\Theta_0,$$

$$\tilde{\sigma}_{33} = 10\sigma_{33}(0, z)/E_r\alpha\Theta_0,$$

$\tilde{\eta} = 10^{-3}\eta(L/2, z)/Er\alpha^2\Theta_0$, $z = \theta_3/h$, (73)

where $S = R/h$ is the slenderness ratio; $E_r = 6.9 \text{ GPa}$, $\alpha = 35.6 \times 10^{-6} \text{ K}^{-1}$, $k_t = 36.42 \text{ W/mK}$, and $d_e = 374 \times 10^{-12} \text{ m/V}$ are the representative moduli of the shell.

Tables 2 and 3 show the results of the convergence study due to increasing the number of SaS. As can be seen, the SaS shell formulation gives an opportunity to calculate the basic FG shell variables with a specified accuracy (six right digits) by using from 3 to 13 SaS inside each layer. Figures 2–5 display the

Figure 2. Through-thickness distributions of the temperature, electric potential, and displacements for a FG four-layer anisotropic cylindrical shell with $R/h = 2$ and $l_1 = l_2 = l_3 = l_4 = 0$, exact solution (25) on the basis of the SaS formulation (4).
distributions of the temperature, electric potential, displacements, heat flux, electric displacement, stresses, and entropy through the thickness of the FG hybrid cylindrical shell for two values of the slenderness ratio employing nine SaSs for each layer. A comparison with the analytical solution based on the SaS formulation for the angle-ply cylindrical shell covered with PZT-5A layers [25] is also presented. These results demonstrate convincingly the high potential of the developed FG thermopiezoelectric shell formulation because the boundary conditions on bottom and top surfaces for the transverse stresses and electric displacement, and the continuity conditions for the heat flux, transverse stresses, and electric displacement at interfaces are satisfied exactly.

8. Analytical solution for FG orthotropic cylindrical panels

In this section, we consider a FG layered orthotropic cylindrical panel subjected to thermal and electromechanical loads. It is assumed that the middle surface of the shell is described by axial and circumferential coordinates $\theta_1$ and $\theta_2$. The boundary conditions for a simply supported cylindrical panel with electrically grounded edges maintained at the reference temperature are written as:

$$
\Theta^{(n)} = \Phi^{(n)} = \sigma_{11}^{(n)} = u_2^{(n)} = u_3^{(n)} = 0 \text{ at } \theta_1 = 0 \text{ and } \theta_1 = a, \\
\Theta^{(n)} = \Phi^{(n)} = u_1^{(n)} = \sigma_{22}^{(n)} = u_3^{(n)} = 0 \text{ at } \theta_2 = 0 \text{ and } \theta_2 = b.
$$

(74)
Figure 4. Through-thickness distributions of the temperature, electric potential, and displacements for a FG four-layer anisotropic cylindrical shell with $R/h = 10$ and $l_1 = l_2 = l_3 = 9$; exact solution [2] on the basis of the SaS formulation ($\circ$).

where $a$ is the length of the panel; $b$ is the length of the circular arc; and $\phi$ is the arc angle (see Figure 6). To satisfy boundary conditions (74), we search the analytical solution by a method of the double Fourier series expansion:

$$\Theta^{(n)j_1}_{rs} = \sum_{r,s} \Theta^{(n)j_1}_{rs} \sin \frac{r \pi \theta_1}{a} \sin \frac{s \pi \theta_2}{b},$$

(75)

$$\varphi^{(n)j_1}_{rs} = \sum_{r,s} \varphi^{(n)j_1}_{rs} \sin \frac{r \pi \theta_1}{a} \sin \frac{s \pi \theta_2}{b},$$

(76)

$$u_1^{(n)j_1}_{rs} = \sum_{r,s} u_1^{(n)j_1}_{rs} \cos \frac{r \pi \theta_1}{a} \sin \frac{s \pi \theta_2}{b},$$

$$u_2^{(n)j_1}_{rs} = \sum_{r,s} u_2^{(n)j_1}_{rs} \cos \frac{r \pi \theta_1}{a} \cos \frac{s \pi \theta_2}{b},$$

$$u_3^{(n)j_1}_{rs} = \sum_{r,s} u_3^{(n)j_1}_{rs} \sin \frac{r \pi \theta_1}{a} \sin \frac{s \pi \theta_2}{b},$$

(77)

where $r$ and $s$ are the wave numbers in $\theta_1$ and $\theta_2$ directions. The external electromechanical loads are also expanded in double Fourier series.

Substituting Fourier series (75)–(77) and Fourier series corresponding to the case of electromechanical loading in Eqs. (11)–(13), (21), (22), (29), (30), (34), (37), (42), (47), (58)–(60), one obtains:

$$J = \sum_{r,s} J_{rs} (\Theta^{(n)j_1}_{rs}),$$

(78)

$$\Pi = \sum_{r,s} \Pi_{rs} (\Theta^{(n)j_1}_{rs}, \varphi^{(n)j_1}_{rs}, u^{(n)j_1}_{rs}).$$

(79)

Invoking variational equations (31) and (39), we arrive at two systems of linear algebraic equations:

$$\frac{\partial J_{rs}}{\partial \Theta^{(n)j_1}_{rs}} = 0,$$

(80)

$$\frac{\partial \Pi_{rs}}{\partial \Theta^{(n)j_1}_{rs}} = 0, \quad \frac{\partial \Pi_{rs}}{\partial u^{(n)j_1}_{rs}} = 0$$

(81)

of orders $K$ and $4K$, where $K = \sum_{n} I_n - N + 1$. The linear systems (80) and (81) are solved independently by a Gaussian elimination method.

The described algorithm was performed with the Symbolic Math Toolbox of MATLAB. This technique gives an opportunity to obtain the analytical solutions for FG thermoelectroelastic orthotropic cylindrical panels with a specified accuracy in the framework of the SaS shell formulation, which asymptotically approach the 3D exact solutions of thermopiezoelectricity as $I_n \to \infty$.

8.1. FG piezoelectric square plate under temperature loading

Here, we study a FG piezoelectric plate polarized in the thickness direction. The plate is loaded by the sinusoidally distributed temperature on the top surface, whereas the bottom surface is
maintained at the reference temperature. The boundary conditions on the top and bottom surfaces are taken to be:

\[ \Theta^+ = \frac{16}{\pi^2} \Theta_0 \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b}, \quad D^+_{ij} = \sigma^+_{ij} = \sigma^+_{33} = 0, \]
\[ \Theta^- = 0, \quad D^-_{ij} = \sigma^-_{ij} = \sigma^-_{33} = \sigma^-_{33} = 0, \] (82)

where \( a = b = 1 \) m, \( \Theta_0 = 1 \) K, and \( T_0 = 293 \) K.

It is supposed that the material constants are distributed in the thickness direction according to the exponential law:

\[ \Xi^{(1)} = \Xi^- e^{\alpha(z+0.5)}, \quad -0.5 \leq z \leq 0.5, \]
\[ \Xi^- = \begin{bmatrix} C_{ijkl}^-, e_{ijk}, \gamma_{ij}, e_{ij}, r_i, k_{ij}, \rho^-, \xi^- \end{bmatrix}, \] (83)
where \( C_{ijkl}^-, e_{ijk}, \gamma_{ij}, e_{ij}, r_i, k_{ij}, \rho^-, \xi^- \) are the values of material constants on the bottom surface, which are considered to be the same as those of cadmium selenide given in [13, 24] and Table 1; \( \alpha \) is the material gradient index defined as:

\[ \alpha = \ln \left( \frac{C_{ijkl}^+}{C_{ijkl}^-} \right), \] (84)

where \( C_{ijkl}^+ \) are the values of elastic constants on the top surface.

To compare the results derived with the exact solution [13] of thermopiezoelectricity, we introduce the scaled basic plate variables as functions of the thickness coordinate:

\[ \tilde{\Theta} = \Theta(P, z), \quad \tilde{q}_1 = q_1(P, z), \quad \tilde{q}_3 = q_3(P, z), \]
\[ \tilde{\eta} = 10^{-4} \times \eta(P, z), \]
\[ \tilde{\varphi} = 10^{-3} \times \varphi(P, z), \quad \tilde{D}_1 = 10^7 \times D_1(P, z), \]
\[ \tilde{D}_3 = 10^8 \times D_3(P, z), \]
\[ \tilde{u}_1 = 10^7 \times u_1(P, z), \quad \tilde{u}_3 = 10^6 \times u_3(P, z), \]
The data listed in Tables 4 and 5 show that the SaS formula permits the solution of the thermal problem for FG piezoelectric thick plates with a very high accuracy using the sufficiently large number of SaS. Here, we take \(e_{23} = 0\) in order to compare the results with the 3D exact solution [13]. It is seen that the SaS technique provides from 12 to 15 right digits for all basic variables at point P for the moderately thick plate with \(a/h = 10\) choosing 15 SaS inside the plate body. However, the use of the larger number of SaS does not improve the accuracy of results. Figures 7–10 display the through-thickness distributions of the temperature, entropy, electric potential, electric displacement, transverse displacement, and stresses for different values of the slenderness ratio \(a/h\) and material gradient index \(\alpha\) employing 11 SaS. The calculations were executed considering the real cadmium selenide with \(e_{23} = -0.138\, \text{C/m}^2\) that corresponds to Table 1. These results demonstrate again the high potential of the proposed FG thermopiezoelectric plate formulation. This is due to the fact that the boundary conditions on the bottom and top surfaces of the plate for the transverse stresses and electric displacement are satisfied correctly.

8.2. FG piezoelectric cylindrical panel under heat flux loading

Finally, we study a cylindrical panel composed of the FG piezoelectric material polarized in the thickness direction. The shell is loaded on the top surface by the sinusoidally distributed heat flux, whereas the bottom surface is assumed to be heat-insulated. The boundary conditions on outer surfaces are:

\[
q_3^+ = q_0 \sin \frac{\pi \theta_1}{a} \sin \frac{\pi \theta_2}{b}, \quad D_3^+ = \sigma_{13}^+ = \sigma_{23}^+ = \sigma_{33}^+ = 0, \quad q_3^- = D_3^- = \sigma_{13}^- = \sigma_{23}^- = \sigma_{33}^- = 0, \tag{86}
\]

where \(q_0 = 1 \, \text{W/m}^2\) and \(T_0 = 293\, \text{K}\). The geometric parameters of the shell are chosen as \(R = 1\, \text{m}, a = 4\, \text{m},\) and \(b = \pi/2\, \text{m}\) (see Figure 6).

### Table 4. Results for a FG square plate with \(a/h = 2\) and \(\alpha = 1\).

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<th>(l_1)</th>
<th>(\hat{\theta}(0.25))</th>
<th>(\phi_{1}(0.25))</th>
<th>(\phi_{2}(0.25))</th>
<th>(\hat{\theta}(0.25))</th>
<th>(\phi_{1}(0.25))</th>
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<td>-23.5920194420774</td>
<td>12.7061733036165</td>
<td>-3.9736308582284</td>
<td>-31.607677279041</td>
</tr>
<tr>
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<td>-6.573916963320</td>
<td>-23.5920194420761</td>
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<td>-31.6076685683050</td>
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<tr>
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<td>-6.573916963344</td>
<td>-23.5920194420761</td>
<td>12.7061747527704</td>
<td>-3.9736308780469</td>
<td>-31.6076685694274</td>
</tr>
</tbody>
</table>

### Table 5. Results for a FG square plate with \(a/h = 10\) and \(\alpha = 1\).

<table>
<thead>
<tr>
<th>(l_1)</th>
<th>(\hat{\theta}(0.25))</th>
<th>(\phi_{1}(0.25))</th>
<th>(\phi_{2}(0.25))</th>
<th>(\hat{\theta}(0.25))</th>
<th>(\phi_{1}(0.25))</th>
<th>(\phi_{2}(0.25))</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.27011930641158</td>
<td>-9.5275942519154</td>
<td>-173.890997876423</td>
<td>4.7905754060072</td>
<td>-1.79679686349016</td>
<td>-9.74393120489477</td>
</tr>
<tr>
<td>7</td>
<td>0.277956102573660</td>
<td>-10.0996183426237</td>
<td>-170.903816003384</td>
<td>4.79429932460303</td>
<td>-1.51540704569816</td>
<td>-2.59079305877469</td>
</tr>
<tr>
<td>11</td>
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<td>-10.0996183426237</td>
<td>-170.903816003384</td>
<td>4.79430922228234</td>
<td>-1.515427722582</td>
<td>-2.59079305877469</td>
</tr>
<tr>
<td>15</td>
<td>0.277956102573660</td>
<td>-10.0996183426237</td>
<td>-170.903816003384</td>
<td>4.79430922228234</td>
<td>-1.515427722582</td>
<td>-2.59079305877469</td>
</tr>
<tr>
<td>Zhong</td>
<td>0.277956102573660</td>
<td>-10.0996183426237</td>
<td>-170.903816003384</td>
<td>4.79430922228234</td>
<td>-1.515427722582</td>
<td>-2.59079305877469</td>
</tr>
</tbody>
</table>

\[\theta_2 = \phi R\]
Figure 7. Through-thickness distributions of the temperature, entropy, electric potential, and electric displacement for a FG piezoelectric square plate with $a/h = 2$ and $l_i = 11$.

Figure 8. Through-thickness distributions of the transverse displacement and stresses for a FG piezoelectric square plate with $a/h = 2$ and $l_i = 11$. 
Figure 9. Through-thickness distributions of the temperature, entropy, electric potential, and electric displacement for a FG piezoelectric square plate with $a/h = 10$ and $I_1 = 11$.

Figure 10. Through-thickness distributions of the transverse displacement and stresses for a FG piezoelectric square plate with $a/h = 10$ and $I_1 = 11$. 
Here, we compare and consider two basic approaches widely used for describing the FG piezoelectric materials, namely, the exponential law (83) and the most popular power law. The latter law reflects a simple rule of mixtures efficiently utilized for finding the effective properties of the FG piezoelectric material and can be presented as follows:

\[
\mathbf{\Xi}^{(1)} = \mathbf{\Xi}^+ V^- + \mathbf{\Xi}^- V^+,
\]

\[
\mathbf{\Xi} = \left[ C_{ijkl}^+, \varepsilon_{ijkl}^+, Y_{ij}^+, \varepsilon_{ij}^+, t_i^+, k_{ij}^+, \rho^+, c_{ij}^+ \right],
\]

where \( V^- (z) \) is the volume fraction defined as:

\[
V^- (z) = (0.5 - z)\beta, \quad -0.5 \leq z \leq 0.5,
\]

where \( \beta \) is the material gradient index.

The material constants on the bottom surface \( \mathbf{\Xi}^- \) are taken to be the same as those of the PZT-5A given in Table 1, whereas the corresponding material constants on the top surface \( \mathbf{\Xi}^+ \) are three times more than on the bottom surface. To investigate the response of the FG piezoelectric rectangular shell more carefully, we consider four values of the material gradient index: \( \alpha = \ln 3 \) in the case of using the exponential law (83), i.e., only one value can be chosen, and \( \beta = 0.2, 2, 5 \) in the case of the power law (87), which allows many values to be taken as illustrated in Figure 11.

To analyze the numerical results effectively, we introduce the dimensionless variables at crucial points as functions of the dimensionless thickness coordinate as follows:

\[
\bar{\Theta} = 10k_r \Theta (a/2, b/2, z)/SRq_0, \quad \bar{\rho}_3 = q_3 (a/2, b/2, z)/q_0,
\]

\[
\bar{\psi} = k_d \psi (a/2, b/2, z)/R^2 \alpha q_0,
\]

\[
\bar{\bar{D}}_3 = k_3 S D_3 (a/2, b/2, z)/R E_t d_4 a_4 q_0,
\]

\[
\bar{\bar{u}}_3 = k_3 u_3 (a/2, b/2, z)/S R^2 \alpha q_0,
\]

\[
\bar{\bar{\sigma}}_{11} = k_3 \sigma_{11} (a/2, b/2, z)/R E_t \alpha q_0,
\]

\[
\bar{\bar{\sigma}}_{22} = k_3 \sigma_{22} (a/2, b/2, z)/R E_t \alpha q_0,
\]

\[
\bar{\bar{\sigma}}_{12} = k_3 \sigma_{12} (0, 0, z)/R E_t \alpha q_0,
\]

\[
\bar{\bar{\sigma}}_{13} = k_3 \sigma_{13} (0, 0, z)/R E_t \alpha q_0,
\]

\[
\bar{\bar{\sigma}}_{23} = k_3 \sigma_{23} (0, 0, z)/R E_t \alpha q_0,
\]

\[
\bar{\bar{\sigma}}_{33} = k_3 \sigma_{33} (0, 0, z)/R E_t \alpha q_0,
\]

\[
\bar{\bar{\eta}} = 10^{-5} k_3 \eta (a/2, b/2, z)/S R E_t \alpha^2 q_0,
\]

Table 8. CPU time (in minutes) by using Intel Core i7-4770 (3.4 GHz, 8 MB Cache) and MATLAB R2011b on a 64-bit Windows platform.

<table>
<thead>
<tr>
<th>( l )</th>
<th>3</th>
<th>7</th>
<th>11</th>
<th>15</th>
<th>19</th>
<th>23</th>
<th>27</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU time</td>
<td>0.17</td>
<td>0.97</td>
<td>3.17</td>
<td>7.25</td>
<td>42.50</td>
<td>59.83</td>
<td>130.38</td>
</tr>
</tbody>
</table>
Figure 12. Through-thickness distributions of the temperature, heat flux, electric potential, and electric displacement for a FG piezoelectric cylindrical panel with $R/h = 4$ and $l_1 = 11$; exact solution [25] on the basis of the SaS formulation (o).

Figure 13. Through-thickness distributions of the transverse displacement and stresses for a FG piezoelectric cylindrical panel with $R/h = 4$ and $l_1 = 11$; exact solution [25] on the basis of the SaS formulation (o).
Figure 14. Through-thickness distributions of the temperature, heat flux, electric potential, and electric displacement for a FG piezoelectric cylindrical panel with $R/h = 10$ and $I_1 = 11$; exact solution [25] on the basis of the SaS formulation (g).

Figure 15. Through-thickness distributions of the transverse displacement and stresses for a FG piezoelectric cylindrical panel with $R/h = 10$ and $I_1 = 11$; exact solution [25] on the basis of the SaS formulation (e).
where $S = R/h$ is the slenderness ratio; $E_t = 10.3 \text{ GPa}$, $\alpha_r = 1.5 \times 10^{-6} \text{ } 1/\text{K}$, $k_r = 1.8 \text{ W/mK}$, and $d_r = 374 \times 10^{-12} \text{ m/V}$ are the representative moduli.

Tables 6 and 7 list the results of the convergence study due to increasing the number of SaS located at Chebyshev polynomial nodes and equispaced nodes, respectively. Note that in the first case the bottom and top surfaces are not included into a set of SaS that permits us to minimize uniformly the error due to Lagrange interpolation. As it turned out, the first formulation provides from 7 to 10 right digits for the basic functions, which are defined via constitutive equations (48)–(50) taking 25 SaS through the thickness of a shell (see Table 6). However, increasing the number of SaS to 27 yields a bit worse results and the choice of 31 SaS leads to the divergence of the symbolic computation algorithm developed. Considering the second formulation with equispaced SaS, one can observe that it provides only three right digits for the heat flux, electric displacement, and transverse stresses at crucial points (see Table 7) and since 27 SaS the divergence occurs. The CPU time required for symbolic computations is given in Table 8.

Figures 12–15 show the distributions of the temperature, heat flux, electric potential, electric displacement, transverse displacement, and stresses through the thickness of the cylindrical panel for two values of the slenderness ratio employing 11 SaS located at Chebyshev polynomial nodes. A comparison with the analytical solution for the homogeneous cylindrical panel composed of the PZT-5A [25] is also given. As can be seen, the boundary conditions on the bottom and top surfaces for transverse components of the heat flux, electric displacement vector, and stress tensor are fulfilled again exactly.

### References


