Exact 3D Thermoelectroelastic Analysis of Piezoelectric Plates through a Sampling Surfaces Method

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The article focuses on the use of the method of sampling surfaces (SaS) to exact three-dimensional (3D) solutions of the steady-state problem of thermoelectroelasticity for piezoelectric laminated plates subjected to thermal loading. The SaS method is based on selecting inside the nth layer $I_n$ not equally spaced SaS parallel to the middle surface of the plate in order to choose temperatures, electric potentials, and displacements of these surfaces as basic plate variables. This permits the representation of the proposed thermopiezoelectric plate formulation in a very compact form. The SaS are located inside each layer at Chebyshev polynomial nodes that improves the convergence of the SaS method significantly. As a result, the SaS method can be applied to 3D exact solutions of thermoelectroelasticity for piezoelectric laminated plates with a specified accuracy using the sufficient number of SaS.

Keywords: thermoelectroelasticity, piezoelectric laminated plate, exact 3D solutions, sampling surfaces method

1. Introduction

Three-dimensional (3D) quasi-static analysis of piezoelectric laminated plates subjected to thermal loading has received considerable attention during the past 20 years [1, 2]. There are at least four approaches to 3D exact solutions of thermoelectroelasticity for piezoelectric plates, namely, the Pagano approach [3–5], the state space approach, the asymptotic approach, and the sampling surfaces (SaS) approach [6, 7]. The first approach was implemented for piezoelectric homogenous and laminated plates in contributions [8–12]. The most popular state space approach was utilized efficiently in [13–16]. The 3D solution of thermoelectroelasticity for piezoelectric rectangular plates using the asymptotic series expansion was obtained by Cheng and Batra [17]. However, the exact 3D solutions for piezoelectric plates subjected to thermal loading on the basis of the SaS technique cannot be found in the open literature. The present article is intended to fill the gap of knowledge in this research area.

The SaS method has been applied very recently to the exact 3D analysis of elastic, electroelastic, and thermoelectroelastic laminated composite plates and shells in papers [18–23]. As SaS denoted by $\Sigma^{0,1}, \Sigma^{0,2}, \ldots, \Sigma^{0,n}$, we choose inner surfaces inside the nth layer and interfaces to introduce temperatures $T^{(0)}, T^{(n)}), \ldots, T^{(n)}, \ldots, T^{(n)}$, electric potentials $\varphi^{(0)}, \varphi^{(n)}, \ldots, \varphi^{(n)}, \ldots, \varphi^{(n)}$, and displacement vectors $u^{(0)}, u^{(n)}, \ldots, u^{(n)}, \ldots, u^{(n)}$, of these surfaces as basic plate variables, where $I_n$ is the total number of SaS of the nth layer ($I_n \geq 3$). Such choice of temperatures, electric potentials, and displacements with the consequent use of Lagrange polynomials of degree $I_n - 1$ in the thickness direction for each layer permits the representation of governing equations of the thermopiezoelectric plate formulation in a very compact form. It is important to mention that the developed approach with the arbitrary number of equally spaced SaS [6] does not work properly with the Lagrange polynomials of high degree because Runge’s phenomenon can occur, which yields the wild oscillation at the edges of the interval when the user deals with any specific functions. If the number of equispaced nodes is increased, then the oscillations become even larger. Fortunately, the use of Chebyshev polynomial nodes [7] can help to improve significantly the behavior of Lagrange polynomials of high degree for which the error will go to zero as $I_n \to \infty$.

The origins of using the SaS can be found in contributions [24, 25] in which three, four, and five equally spaced SaS are utilized. It is interesting to note also that in a finite layer method [26], which is the most efficient semi-analytical method for the 3D analysis of simply supported laminated plates and cylindrical shells [27–30], the structure is divided into a number of finite layers following the general layer-wise concept [31–34]. Within each finite layer, the trigonometric functions are employed for in-plane interpolations of displacements in a displacement-based formulation [30] and additionally transverse stresses in a mixed formulation [29], whereas the lower-order Lagrange polynomials with equispaced nodal points are accepted for the interpolation in the thickness.
direction, i.e., the h-refinement is adopted. Thus, the difference between the SaS method and the finite layer method consists in the following: the p-refinement is used in the former, while the h-refinement is used in the latter. Wu et al. [29, 30] showed that the finite layer method with equally spaced nodal surfaces yields good predictions of the mechanical behavior of composite plates and shells. However, the 3D solutions derived are approximate. To obtain the exact 3D solutions, the p-refinement should be invoked. As pointed out earlier, the SaS method utilizes the Lagrange polynomials of high degree with Chebyshev polynomial nodes that allows one to minimize uniformly the error due to Lagrange interpolation. This fact gives an opportunity to find the exact 3D solutions for thermal laminated composite shells with a prescribed accuracy employing the sufficiently large number of SaS.

The authors restrict themselves to finding five right digits in all examples presented except for section 7.1 with the results of the convergence study. The better accuracy is possible but requires more SaS inside each layer to be taken.

2. Description of Temperature Field

Consider a laminated plate of the thickness \( h \). Let the middle surface \( \Omega \) be described by Cartesian coordinates \( x_1 \) and \( x_2 \). The coordinate \( x_3 \) is oriented in the thickness direction. The transverse coordinates of SaS inside the \( n \)th layer are defined as:

\[
\begin{align*}
  x_3^{(n)-1} &= x_3^{[n-1]}, \quad x_3^{(n)k} = x_3^{[n]}, \\
  x_3^{(n)n} &= \frac{1}{2}(x_3^{[n-1]} + x_3^{[n]}) - \frac{1}{2} h_n \cos \left( \pi \frac{2m_n - 3}{(L_n - 2)} \right),
\end{align*}
\]

where \( x_3^{[n-1]} \) and \( x_3^{[n]} \) are the transverse coordinates of layer interfaces \( \Omega^{[n-1]} \) and \( \Omega^{[n]} \) depicted in Figure 1; \( h_n = x_3^{[n]} - x_3^{[n-1]} \) is the thickness of the \( n \)th layer; \( L_n \) is the number of SaS corresponding to the \( n \)th layer; the index \( n \) identifies the belonging of any quantity to the \( n \)th layer and runs from 1 to \( N \); \( N \) is the total number of layers; the index \( m_n \) identifies the belonging of any quantity to the inner SaS of the \( n \)th layer and runs from 2 to \( L_n - 1 \), whereas the indices \( i_n, j_n, \) and \( k_n \) will be introduced later for describing all SaS of the \( n \)th layer run from 1 to \( L_n \). Besides, the tensorial indices \( i, j, k, l \) range from 1 to 3 and Greek indices \( \alpha, \beta \) range from 1 to 2.

It is worth noting that the transverse coordinates of inner SaS (2) coincide with the coordinates of Chebyshev polynomial nodes [35]. This fact has a great meaning for a convergence of the SaS method [7].

The relation between the temperature \( T \) and the temperature gradient \( \Gamma \) is given by:

\[
\Gamma = \nabla T.
\]

In a component form, it can be written as:

\[
\Gamma_j = T_j,
\]

where the symbol \(( \ldots )_j \) stands for the partial derivatives with respect to coordinates \( x_j \).

We start now with the first assumption of the proposed thermopiezoelectric laminated plate formulation. Let us assume that the temperature and temperature gradient fields are distributed through the thickness of the \( n \)th layer as follows:

\[
\begin{align*}
  T^{(n)} &= \sum_{i_n} L^{(n)i_n} T^{(n)i_n}, \quad x_3^{[n-1]} \leq x_3 \leq x_3^{[n]}, \\
  \Gamma^{(n)}_j &= \sum_{i_n} L^{(n)i_n} \Gamma^{(n)i_n}_j, \quad x_3^{[n-1]} \leq x_3 \leq x_3^{[n]},
\end{align*}
\]

where \( T^{(n)i_n}(x_1, x_2) \) are the temperatures of SaS \( \Omega^{(n)i_n} \) of the \( n \)th layer; \( \Gamma^{(n)i_n}_j(x_1, x_2) \) are the components of the temperature gradient at the same SaS; and \( L^{(n)i_n}(x_3) \) are the Lagrange polynomials of degree \( L_n - 1 \) defined as:

\[
\begin{align*}
  T^{(n)i_n} &= T(x_3^{(n)i_n}), \\
  \Gamma^{(n)i_n}_j &= \Gamma_j(x_3^{(n)i_n}), \\
  L^{(n)i_n}(x_3) &= \prod_{j_n \neq i_n} \frac{x_3 - x_3^{(n)j_n}}{x_3^{(n)i_n} - x_3^{(n)j_n}}.
\end{align*}
\]

The use of relations (4), (5), (7), and (8) yields:

\[
\begin{align*}
  \Gamma^{(n)i_n}_0 &= T^{(n)i_n}, \\
  \Gamma^{(n)i_n}_3 &= \sum_{j_n} M^{(n)j_n}(x_3^{(n)i_n}) T^{(n)i_n}_j,
\end{align*}
\]
Exact 3D Thermoelectroelastic Analysis

where $M^{(n)j_o} = L^{(n)j_o}_3$ are the derivatives of Lagrange polynomials, which are calculated as SaS as follows:

\[
M^{(n)j_o}(x_3^{(n)j_o}) = \frac{1}{x_3^{(n)j_o} - x_3^{(n)k_o}} \prod_{k_o \neq j_o} \frac{x_3^{(n)k_o} - x_3^{(n)k_o}}{x_3^{(n)j_o} - x_3^{(n)k_o}}, \text{for } j_o \neq k_o.
\]

\[
M^{(n)j_o}(x_3^{(n)j_o}) = - \sum_{j_o \neq k_o} M^{(n)j_o}(x_3^{(n)j_o}). \tag{12}
\]

It is seen from Eq. (11) that the transverse component of the temperature gradient $\Gamma_3^{(n)j_o}$ is represented as a linear combination of temperatures of all SaS of the nth layer $\Gamma^{(n)j_o}$.

3. Description of Electric Field

The relation between the electric potential $\varphi$ and the electric field $E$ is given by:

\[
E = - \nabla \varphi. \tag{13}
\]

In a component form, it is expressed as:

\[
E_i = - \varphi_{,i}. \tag{14}
\]

Following the SaS technique, we introduce now the second assumption of the thermopiezoelectric laminated plate formulation. Let the electric potential and the electric field vector be distributed through the thickness of the nth layer similar to temperature distributions (5) and (6), that is,

\[
\varphi^{(n)} = \sum_{i_o} L^{(n)j_o} \varphi^{(n)j_o}, \quad x_3^{[n-1]} \leq x_3 \leq x_3^{[n]}, \tag{15}
\]

\[
E_i^{(n)} = \sum_{i_o} L^{(n)j_o} E_i^{(n)j_o}, \quad x_3^{[n-1]} \leq x_3 \leq x_3^{[n]}, \tag{16}
\]

where $\varphi^{(n)j_o}(x_1, x_2)$ are the electric potentials of SaS of the nth layer; $E_i^{(n)j_o}(x_1, x_2)$ are the components of the electric field vector at the same SaS defined as:

\[
\varphi^{(n)j_o} = \varphi(x_3^{(n)j_o}), \tag{17}
\]

\[
E_i^{(n)j_o} = E_i(x_3^{(n)j_o}). \tag{18}
\]

The use of Eqs. (14), (15), (17), and (18) yields:

\[
E_i^{(n)j_o} = - \varphi_{,j_o}, \tag{19}
\]

\[
E_i^{(n)j_o} = - \sum_{j_o} M^{(n)j_o}(x_3^{(n)j_o}) \varphi^{(n)j_o}. \tag{20}
\]

One can see that Eq. (20) is similar to Eq. (11). Thus, the transverse component of the electric field $E_3^{(n)j_o}$ is represented as a linear combination of electric potentials of SaS of the nth layer $\varphi^{(n)j_o}$.

4. Description of Mechanical Field

The strain components $\varepsilon_{ij}$ are written as:

\[
2\varepsilon_{ij} = u_{i,j} + u_{j,i}, \tag{21}
\]

where $u_i$ are the displacements of the plate.

Following the SaS technique, we introduce the third assumption of the thermopiezoelectric laminated plate formulation developed. Let us assume that displacement and strain distributions through the thickness of the nth layer are similar to thermal and electric field distributions (5), (6), (15), and (16). Thus, we have:

\[
u_i^{(n)j_o} = \sum_{i_o} L^{(n)j_o} u_i^{(n)j_o}, \quad x_3^{[n-1]} \leq x_3 \leq x_3^{[n]}, \tag{22}
\]

\[
\varepsilon_{ij}^{(n)j_o} = \sum_{j_o} L^{(n)j_o} \varepsilon_{ij}^{(n)j_o}, \quad x_3^{[n-1]} \leq x_3 \leq x_3^{[n]}, \tag{23}
\]

where $u_i^{(n)j_o}(x_1, x_2)$ are the displacements of SaS $\Omega^{(n)j_o}$; $\varepsilon_{ij}^{(n)j_o}(x_1, x_2)$ are the strains of the same SaS defined as:

\[
u_i^{(n)j_o} = u_i(x_3^{(n)j_o}), \tag{24}
\]

\[
\varepsilon_{ij}^{(n)j_o} = \varepsilon_{ij}(x_3^{(n)j_o}). \tag{25}
\]

Using Eqs. (21), (22), (24), and (25), one obtains:

\[
2 \varepsilon_{i\alpha}^{(n)j_o} = u_{i,\alpha}^{(n)j_o} + u_{\alpha,i}^{(n)j_o}, \tag{26}
\]

\[
2 \varepsilon_{i\alpha}^{(n)j_o} = \beta_{\alpha,i}^{(n)j_o} + \beta_{i,\alpha}^{(n)j_o}, \tag{27}
\]

\[
\varepsilon_{33}^{(n)j_o} = \beta_{33}^{(n)j_o}, \tag{28}
\]

\[
\varepsilon_{i3}^{(n)j_o} = u_{i,3}^{(n)j_o}, \tag{29}
\]

where $\beta_{ij}^{(n)j_o}(x_1, x_2)$ are the values of derivatives of displacements with respect to transverse coordinate at SaS defined as:

\[
\beta_{ij}^{(n)j_o} = \sum_{j_o} M^{(n)j_o}(x_3^{(n)j_o}) u_i^{(n)j_o}. \tag{30}
\]

This means that the key functions $\beta_{ij}^{(n)j_o}$ of the proposed thermopiezoelectric laminated plate formulation are represented as a linear combination of displacements of SaS of the nth layer $u_i^{(n)j_o}$.

5. Variational Formulation of Heat Conduction Problem

The variational equation for the laminated plate can be written as:

\[
\delta J = 0, \tag{31}
\]
where $J$ is the basic functional of the heat conduction theory given by:

$$J = \frac{1}{2} \int \int_{\Omega} \sum_{n} \int_{x_{3}^{[n-1]}}^{x_{3}^{[n]}} q_{n}^{(n)} \Gamma_{i}^{(n)} \, dX_{1} \, dX_{2} \, dX_{3}$$

$$- \int \int_{\Omega} \left[ \dot{q}_{n} T + \frac{1}{2} \dot{T} \right] \, d\Omega,$$

(32)

where $q_{n}^{(n)}$ is the heat flux vector of the $n$th layer; $\dot{q}_{n}$ is the specified heat flux on the boundary surface $\dot{\Omega}$; $\dot{T}$ is the convective heat transfer coefficient; $\dot{T}$ is the reference temperature for convective transfer; and $\dot{\Omega}$ is a part of outer surfaces.

Substituting the temperature gradient distribution (6) into functional (32) and introducing heat flux resultants:

$$Q_{n}^{(n)\dot{e}_{n}} = \int_{x_{3}^{[n-1]}}^{x_{3}^{[n]}} q_{n}^{(n)} L^{(n)\dot{e}_{n}} \, dX_{3},$$

(33)

one finds that

$$J = \frac{1}{2} \int \int_{\Omega} \sum_{n} \sum_{i} \int_{x_{3}^{[n-1]}}^{x_{3}^{[n]}} Q_{n}^{(n)\dot{e}_{n}} \Gamma_{i}^{(n)\dot{e}_{n}} \, dX_{1} \, dX_{2}$$

$$- \int \int_{\Omega} \left[ \dot{q}_{n} T + \frac{1}{2} \dot{\mu} (T - \dot{T}) \right] \, d\Omega.$$

(34)

Now, we accept the forth assumption of the proposed thermopiezoelectric laminated plate formulation. Let the constitutive equations be the Fourier’s heat conduction equations:

$$q_{i}^{(n)} = -k_{i}^{(n)} \dot{T}^{(n)}, x_{3}^{[n-1]} \leq x_{3} \leq x_{3}^{[n]},$$

(35)

where $k_{i}^{(n)}$ are the components of the thermal conductivity tensor of the $n$th layer.

Substituting the constitutive Eqs. (35) into Eq. (33) and accounting for the through-the-thickness distribution (6), we obtain:

$$Q_{i}^{(n)\dot{e}_{n}} = -\sum_{j} \Lambda^{(n)\dot{e}_{n,j}} k_{i j}^{(n)} \dot{T}^{(n)\dot{e}_{n,j}},$$

(36)

where

$$\Lambda^{(n)\dot{e}_{n,j}} = \int_{x_{3}^{[n-1]}}^{x_{3}^{[n]}} L^{(n)\dot{e}_{n}} L^{(n)\dot{e}_{n,j}} \, dX_{3},$$

(37)

6. Variational Formulation of Thermoelectroelastic Problem

The variational equation for the thermoelectroelastic laminated plate in the case of conservative loading can be written as:

$$\delta \Pi = 0,$$

(38)

where $\Pi$ is the basic functional of the theory of thermopiezoelectricity given by:

$$\Pi = \frac{1}{2} \int \int_{\Omega} \sum_{n} \sum_{i} \int_{x_{3}^{[n-1]}}^{x_{3}^{[n]}} \left( \epsilon_{i j}^{(n)\dot{e}_{n,j}} - D_{i}^{(n)\dot{e}_{n,j}} \right) \, dX_{1} \, dX_{2} \, dX_{3}$$

$$\times \delta \Pi = 0,$$

(39)

$$W = \int \int_{\Omega} \left( p_{i}^{[N]} u_{i}^{[N]} - p_{i}^{[U]} u_{i}^{[U]} - Q^{-} \varphi^{[N]} - Q^{+} \varphi^{[U]} \right) \, dX_{1} \, dX_{2} \, dX_{3} + W_{\Sigma},$$

(40)

where $\varphi^{[N]}$ is the stress tensor of the $n$th layer; $D_{i}^{(n)\dot{e}_{n,j}}$ is the electric displacement vector of the $n$th layer; $\eta^{(n)\dot{e}_{n,j}}$ is the entropy density of the $n$th layer; $u_{i}^{[0]} = u_{i}^{[0]}$ and $u_{i}^{[N]} = u_{i}^{[N]}$ are the displacements of bottom and top surfaces $\Omega^{[0]}$ and $\Omega^{[N]}$; $\varphi^{[0]} = \varphi^{[0]}$ and $\varphi^{[N]} = \varphi^{[N]}$ are the electric potentials of bottom and top surfaces; $p_{i}^{[N]}$ and $p_{i}^{[U]}$ are the loads acting on outer surfaces; $Q^{-}$ and $Q^{+}$ are the specified electric charges on outer surfaces; $W_{\Sigma}$ is the work done by external loads applied to the edge surface $\Sigma$; $\varphi^{(n)\dot{e}_{n,j}}$ is the temperature rise from the initial reference temperature $T_{0}$ defined as:

$$\varphi^{(n)\dot{e}_{n,j}} = T^{(n)} - T_{0}.$$

(41)

Substituting the electric field and strain distributions (16) and (23) and the temperature distribution:

$$\varphi^{(n)\dot{e}_{n,j}} = \sum_{x_{3}} L^{(n)\dot{e}_{n,j}}, x_{3}^{[n-1]} \leq x_{3} \leq x_{3}^{[n]},$$

(42)

$$\varphi^{(n)\dot{e}_{n,j}} = \Theta(x_{3}^{[n]}) \dot{e}_{n,j},$$

(43)

which follows directly from Eqs. (5), (7), and (41) into functional (39), and introducing stress resultants:

$$H_{i}^{(n)\dot{e}_{n,j}} = \int_{x_{3}^{[n-1]}}^{x_{3}^{[n]}} D_{i}^{(n)\dot{e}_{n,j}} \, dX_{3},$$

(44)

electric displacement resultants:

$$K_{i}^{(n)\dot{e}_{n,j}} = \int_{x_{3}^{[n-1]}}^{x_{3}^{[n]}} E_{i}^{(n)\dot{e}_{n,j}} \, dX_{3},$$

(45)

and entropy resultants:

$$S^{(n)\dot{e}_{n,j}} = \int_{x_{3}^{[n-1]}}^{x_{3}^{[n]}} \eta \dot{e}_{n,j} \, dX_{3},$$

(46)

one obtains:

$$\delta \Pi = 0,$$

(38)
Finally, we introduce the fifth assumption of the thermopiezoelectric laminated plate formulation developed. Let us consider the case of linear thermoelastic materials. Therefore, the constitutive equations [31] are written as follows:

\[ \sigma_{ij}^{(n)} = c_{ijkl}^{(n)} e_{kl}^{(n)} - c_{ijkl}^{(e)} E_{k}^{(n)} - \gamma_{ij}^{(n)} \Theta^{(n)} , \quad x_3^{[n-1]} \leq x_3^{[n]} \leq x_3^{[n]} , \]  
\[ D_{ij}^{(n)} = e_{ijkl}^{(n)} + e_{ijkl}^{(e)} E_{k}^{(n)} + \epsilon_{ij}^{(n)} \Theta^{(n)} , \quad x_3^{[n-1]} \leq x_3^{[n]} , \]  
\[ \eta^{(n)} = \gamma_{kl}^{(n)} e_{kl}^{(n)} + r_{ik}^{(n)} E_{k}^{(n)} + \chi^{(n)} \Theta^{(n)} , \quad x_3^{[n-1]} \leq x_3^{[n]} , \]

where \( c_{ijkl}^{(n)} \) are the elastic constants of the \( n \)th layer; \( e_{ijkl}^{(n)} \) are the piezoelectric constants of the \( n \)th layer; \( \gamma_{ij}^{(n)} \) are the thermal stress coefficients of the \( n \)th layer; \( \epsilon_{ij}^{(n)} \) are the dielectric constants of the \( n \)th layer; \( r_{ik}^{(n)} \) are the pyroelectric constants of the \( n \)th layer; \( \chi^{(n)} \) is the entropy-temperature coefficient of the \( n \)th layer defined as:

\[ \chi^{(n)} = \rho^{(n)} e_{v}^{(n)} / T_0 , \]

where \( \rho^{(n)} \) and \( e_{v}^{(n)} \) are the mass density and specific heat per unit mass at constant volume.

The use of through-the-thickness distributions (16), (23), and (42) in Eqs. (44)–(46) and Eqs. (48)–(50) yields:

\[ H_{ij}^{(n)} = \sum J_{i} A_j^{(n)} , \]
\[ R_{i}^{(n)} = \sum J_{i} A_j^{(n)} , \]
\[ S_{ij}^{(n)} = \sum J_{i} A_j^{(n)} , \]

7. 3D Solution for Piezoelectric Plate in Cylindrical Bending

In this section, we study a piezoelectric laminated plate in thermal cylindrical bending. The boundary conditions for the simply supported plate whose edges are electrically grounded and maintained at the reference temperature can be written as:

\[ \Theta^{(n)} = \phi^{(n)} = 0 \quad \text{at} x_1 = 0 \quad \text{and} x_1 = a , \]

\[ \sigma_{11}^{(n)} = u_{2}^{(n)} = u_{3}^{(n)} = 0 \quad \text{at} x_1 = 0 \quad \text{and} x_1 = a , \]

where \( a \) is the width of the plate. To satisfy boundary conditions, we search the analytical solution of the problem by a method of Fourier series expansion:

\[ \Theta^{(n)} = \sum_{r} \Theta_{r}^{(n)} \sin \frac{r \pi x_1}{a} , \]
\[ \phi^{(n)} = \sum_{r} \phi_{r}^{(n)} \sin \frac{r \pi x_1}{a} , \]

where \( r \) is the wave number along the \( x_1 \)-direction.

Substituting a Fourier series (57) in Eq. (34) and using relations (10), (11), (36), (41), and (43), one obtains:

\[ J = \sum_{r} J_{r} (\Theta_{r}^{(n)} u_{r}) . \]

Invoking the variational equation (31), we arrive at the system of linear algebraic equations:

\[ \frac{\partial J_{r}}{\partial \Theta_{r}^{(n)} u_{r}} = 0 \]

of order \( K \), where \( K = \sum_{n} = N + 1 \). Thus, the temperatures \( \Theta_{r}^{(n)} \) of SaS of the \( n \)th layer can be easily found by using a method of Gaussian elimination.

Substituting further Fourier series (57), (58), and (59) in Eq. (47) and allowing for relations (19), (20), (26)–(28), (30),

<table>
<thead>
<tr>
<th>Table 1. Properties of piezoelectric materials</th>
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<tbody>
<tr>
<td>Material</td>
</tr>
<tr>
<td>( C_{1111} ), GPa</td>
</tr>
<tr>
<td>( C_{2222} ), GPa</td>
</tr>
<tr>
<td>( C_{3333} ), GPa</td>
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<tr>
<td>( \gamma_{11} ), Pa/K</td>
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<td>( \gamma_{22} ), Pa/K</td>
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<tr>
<td>( \gamma_{33} ), Pa/K</td>
</tr>
<tr>
<td>( e_{311} ), C/m²</td>
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<td>( e_{322} ), C/m²</td>
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<td>( e_{33} ), F/m</td>
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<tr>
<td>( r_{3} ), C/m²K</td>
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<td>( k_{11} ), W/m/K</td>
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<tr>
<td>( \rho ), Kg/m³</td>
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<tr>
<td>( c_{v} ), J/KgK</td>
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</tbody>
</table>
and (52)-(54), we find:

$$\Pi = \sum_r \Pi_r (u^{(n)}_{tr}, u^{(n)}_{3r}, \varphi^{(n)}_r, \Theta^{(n)}_r).$$  \hspace{1cm} (62)

The use of Eqs. (38) and (62) leads to a system of linear algebraic equations:

$$\frac{\partial \Pi_r}{\partial u^{(n)}_{tr}} = 0, \quad \frac{\partial \Pi_r}{\partial u^{(n)}_{3r}} = 0, \quad \frac{\partial \Pi_r}{\partial \varphi^{(n)}_r} = 0$$  \hspace{1cm} (63)

of order $3K$. The linear system (63) is solved through a method of Gaussian elimination.

The described algorithm was performed with the Symbolic Math Toolbox, which incorporates symbolic computations into the numeric environment of MATLAB. This in turn gives the possibility to derive the exact solutions of thermoelectroelasticity for laminated anisotropic plates in cylindrical bending with a specified accuracy.

### 7.1. Validation of SaS Approach for Heat Flux Boundary Conditions

As a numerical example, we consider a simply supported single-layer plate composed of cadmium selenide polarized in the thickness direction. The material properties are presented in [8, 36] and Table 1. Let the plate be loaded on the top surface by the sinusoidally distributed heat flux, whereas the bottom surface is assumed to be heat-insulated. The boundary conditions on the top and bottom surfaces are taken to be:

$$q_1^{(1)} = q_0 \sin \frac{\pi x}{a}, \quad D_3^{(1)} = \sigma_{13}^{(1)} = \sigma_{31}^{(1)} = \sigma_{33}^{(1)} = 0 \text{ at } x_3 = h/2,$$

$$q_3^{(1)} = 0, \quad D_3^{(1)} = \sigma_{13}^{(1)} = \sigma_{31}^{(1)} = \sigma_{33}^{(1)} = 0 \text{ at } x_3 = -h/2.$$  \hspace{1cm} (64)

### 7.2. Validation of SaS Approach for Convective Boundary Conditions

Next, we study the same cadmium selenide plate. To validate this type of thermal loading, we consider boundary conditions on the top and bottom surfaces [8] as follows:

$$\Theta^{(1)}_{x_3} + h^+ \Theta^{(1)} = h^+ \Theta_0 \sin \frac{\pi x_1}{a},$$

$$D_3^{(1)} = \sigma_{13}^{(1)} = \sigma_{31}^{(1)} = \sigma_{33}^{(1)} = 0 \text{ at } x_3 = h/2,$$

$$\Theta^{(1)}_{x_3} - h^- \Theta^{(1)} = 0,$$

$$D_3^{(1)} = \sigma_{13}^{(1)} = \sigma_{31}^{(1)} = \sigma_{33}^{(1)} = 0 \text{ at } x_3 = -h/2.$$  \hspace{1cm} (66)

### Table 2. Results of the convergence study for a single-layer piezoelectric plate with $a/h = 2$ in the case of heat flux boundary conditions

<table>
<thead>
<tr>
<th>$I_1$</th>
<th>$\bar{\Theta}(0.5)$</th>
<th>$\bar{\varphi}(0.5)$</th>
<th>$\bar{u}_1(0.5)$</th>
<th>$\bar{u}_3(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.514830974804894</td>
<td>-1.276585132031578</td>
<td>-6.992090213777479</td>
<td>4.0394724721985</td>
</tr>
<tr>
<td>7</td>
<td>1.51658380698255</td>
<td>-1.287169963586866</td>
<td>-6.93622836092725</td>
<td>4.04592423554766</td>
</tr>
<tr>
<td>11</td>
<td>1.516583806982569</td>
<td>-1.287169966676169</td>
<td>-6.936228359388562</td>
<td>4.0459242194983</td>
</tr>
<tr>
<td>15</td>
<td>1.516583806982570</td>
<td>-1.287169966676170</td>
<td>-6.936228359388563</td>
<td>4.04592421942870</td>
</tr>
<tr>
<td>19</td>
<td>1.516583806982570</td>
<td>-1.287169966676169</td>
<td>-6.936228359388563</td>
<td>4.04592421942869</td>
</tr>
<tr>
<td>23</td>
<td>1.516583806982570</td>
<td>-1.287169966676170</td>
<td>-6.936228359388563</td>
<td>4.04592421942869</td>
</tr>
<tr>
<td>27</td>
<td>1.516583806982570</td>
<td>-1.287169966676170</td>
<td>-6.936228359388563</td>
<td>4.04592421942869</td>
</tr>
</tbody>
</table>

where $a = 1\text{m}$, $q_0 = 1\text{W/m}^2$, and $T_0 = 293\text{K}$. To evaluate the results of the convergence study, we introduce the dimensionless field variables:

$$\bar{\Theta} = 10h_0 a^2 \Theta(a/2, z), \quad \bar{\varphi} = 10^3 k_0 a^2 q_0 \varphi(a/2, z),$$

$$\bar{u}_1 = 100h_0 a^2 q_0 \Theta_0 a z, \quad \bar{u}_3 = 100h_0 a^2 q_0 \Theta_0 a z.$$  \hspace{1cm} (65)

### Table 3. Results of the convergence study for a single-layer piezoelectric plate with $a/h = 10$ in the case of heat flux boundary conditions

<table>
<thead>
<tr>
<th>$I_1$</th>
<th>$\bar{\Theta}(0.5)$</th>
<th>$\bar{\varphi}(0.5)$</th>
<th>$\bar{u}_1(0.5)$</th>
<th>$\bar{u}_3(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.0353377086088057</td>
<td>-1.329963076735578</td>
<td>-4.750069515592858</td>
<td>4.830450207594048</td>
</tr>
<tr>
<td>7</td>
<td>1.0353377187913652</td>
<td>-1.329961621213969</td>
<td>-4.7499419430288</td>
<td>4.829776282323792</td>
</tr>
<tr>
<td>11</td>
<td>1.0353377187913652</td>
<td>-1.329961621213969</td>
<td>-4.7499419430288</td>
<td>4.829776282323684</td>
</tr>
<tr>
<td>15</td>
<td>1.0353377187913653</td>
<td>-1.329961621213969</td>
<td>-4.7499419430288</td>
<td>4.829776282323684</td>
</tr>
<tr>
<td>19</td>
<td>1.0353377187913652</td>
<td>-1.329961621213969</td>
<td>-4.7499419430288</td>
<td>4.829776282323683</td>
</tr>
</tbody>
</table>
where \( a = 1 \text{m}, \Theta_0 = 1 \text{K}, T_0 = 293 \text{K}, hh^- = 0.2, \) and \( hh^+ = 2.\)

To evaluate the results of the convergence study, the following dimensionless variables are introduced:

\[
\bar{\Theta} = \Theta(a/2, z)/\Theta_0, \quad \bar{q}_3 = 10aq_3(a/2, z)/S\kappa_0\Theta_0, \\
\bar{\varphi} = 10^3d_\varphi(a/2, z)/h\kappa_0\Theta_0, \\
\bar{D}_3 = S^2D_3(a/2, z)/d_0E_\varphi\kappa_0\Theta_0, \\
\bar{u}_1 = 10u_1(0, z)/S\kappa_0\Theta_0, \quad \bar{u}_3 = 100u_3(a/2, z)/S^2\kappa_0\Theta_0, \\
\bar{\sigma}_{11} = 10^3S^2\sigma_{11}(a/2, z)/E_\varphi\kappa_0\Theta_0, \\
\bar{\sigma}_{13} = 10^3S^2\sigma_{13}(0, z)/E_\varphi\kappa_0\Theta_0, \\
\bar{\eta} = 10^{-3}\eta(a/2, z)/E_\varphi\kappa_0^3\Theta_0, \quad z = x_3/h. 
\]

where \( S = a/h \) is the slenderness ratio. The reference material data are taken to be \( \kappa_0 = 4.396 \times 10^{-6} \text{K}, \) \( k_0 = 9 \text{W/mK}, \) \( d_0 = 3.9238 \times 10^{-12} \text{C/N}, \) and \( E_0 = 42.785 \text{GPa}.\)

Tables 4 and 5 list the results of the convergence study utilizing a various number of SaS \( I_1 \) inside the plate body. A comparison with the 3D analytical solution of Dube et al. [8] is also given. The derived results demonstrate convincingly the high potential of the developed thermopiezoelectric plate formulation. It is necessary to mention that due to consideration of boundary conditions (66) the bottom and top surfaces are included into a set of SaS. Figure 2 displays through-the-thickness distributions of the temperature, heat flux, electric potential, electric displacement, transverse displacement, transverse stresses, and entropy for different slenderness ratios \( a/h \) employing 11 SaS. As can be seen, the boundary conditions on the bottom and top surfaces for transverse components of the electric displacement vector and stress tensor are satisfied exactly by using the constitutive equations (48) and (49).

### Table 4. Results of the convergence study for a single-layer piezoelectric plate with \( a/h = 2 \) in the case of convective boundary conditions

<table>
<thead>
<tr>
<th>( I_1 )</th>
<th>( \bar{\Theta}(-0.5) )</th>
<th>( \bar{\Theta}(0.5) )</th>
<th>( \bar{\varphi}(0.5) )</th>
<th>( \bar{u}_1(0.5) )</th>
<th>( \bar{u}_3(0.5) )</th>
<th>( \bar{\sigma}_{11}(0.5) )</th>
<th>( \bar{\sigma}_{13}(-0.25) )</th>
<th>( \bar{\sigma}_{33}(0) )</th>
<th>( \bar{D}_3(0) )</th>
<th>( \bar{q}_3(0.5) )</th>
<th>( \bar{\eta}(0.5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.28881</td>
<td>0.63570</td>
<td>-5.2859</td>
<td>-2.9088</td>
<td>9.9580</td>
<td>-15.682</td>
<td>3.1848</td>
<td>-2.8412</td>
<td>-1.3398</td>
<td>-10.920</td>
<td>3.9685</td>
</tr>
<tr>
<td>7</td>
<td>0.28881</td>
<td>0.63570</td>
<td>-5.2859</td>
<td>-2.9087</td>
<td>9.9580</td>
<td>-13.098</td>
<td>3.6968</td>
<td>3.9052</td>
<td>-1.3255</td>
<td>-10.929</td>
<td>3.9685</td>
</tr>
<tr>
<td>9</td>
<td>0.28881</td>
<td>0.63570</td>
<td>-5.2859</td>
<td>-2.9087</td>
<td>9.9580</td>
<td>-13.089</td>
<td>3.6938</td>
<td>3.8899</td>
<td>-1.3256</td>
<td>-10.929</td>
<td>3.9685</td>
</tr>
<tr>
<td>11</td>
<td>0.28881</td>
<td>0.63570</td>
<td>-5.2859</td>
<td>-2.9087</td>
<td>9.9580</td>
<td>-13.089</td>
<td>3.6938</td>
<td>3.8898</td>
<td>-1.3256</td>
<td>-10.929</td>
<td>3.9685</td>
</tr>
<tr>
<td>13</td>
<td>0.28881</td>
<td>0.63570</td>
<td>-5.2859</td>
<td>-2.9087</td>
<td>9.9580</td>
<td>-13.089</td>
<td>3.6938</td>
<td>3.8898</td>
<td>-1.3256</td>
<td>-10.929</td>
<td>3.9685</td>
</tr>
</tbody>
</table>
| Dube | 0.2888 | 0.6357 | -5.286 | -2.909 | 9.958 | -13.09 | -

### 8. 3D Solution for Piezoelectric Rectangular Plate

Here, we consider a piezoelectric laminated rectangular plate subjected to thermal loading. The boundary conditions for the simply supported plate whose edges are electrically grounded and maintained at the reference temperature are written as:

\[
\Theta^{(a)} = \varphi^{(a)} = 0 = u_1^{(a)} = u_3^{(a)} = 0 \quad \text{at} \ x_1 = 0 \quad \text{and} \ x_1 = a, \\
\Theta^{(a)} = \varphi^{(a)} = 0 = u_1^{(a)} = u_3^{(a)} = 0 \quad \text{at} \ x_2 = 0 \quad \text{and} \ x_2 = b, 
\]

where \( a \) and \( b \) are the plate dimensions. To satisfy boundary conditions, we search the analytical solution of the problem by a method of double Fourier series expansion:

\[
\Theta^{(r,s)} = \sum_{r,s} \Theta^{(r,s)} \sin \frac{r \pi x_1}{a} \sin \frac{s \pi x_2}{b}, \\
\varphi^{(r,s)} = \sum_{r,s} \varphi^{(r,s)} \sin \frac{r \pi x_1}{a} \sin \frac{s \pi x_2}{b}, \\
u_1^{(r,s)} = \sum_{r,s} u_1^{(r,s)} \cos \frac{r \pi x_1}{a} \sin \frac{s \pi x_2}{b}, \\
u_2^{(r,s)} = \sum_{r,s} u_2^{(r,s)} \cos \frac{r \pi x_1}{a} \sin \frac{s \pi x_2}{b}, \\
u_3^{(r,s)} = \sum_{r,s} u_3^{(r,s)} \sin \frac{r \pi x_1}{a} \sin \frac{s \pi x_2}{b},
\]

where \( r \) and \( s \) are the wave numbers in plane directions.

Using Fourier series (69), (70), and (71) in Eqs. (34) and (47) and accounting for relations (10), (11), (19), (20), (26), (27), (28), (30), (36), (41), (43), (52), (53), and (54), one derives:

\[
J = \sum_{r,s} J_{rs} \left( \Theta^{(r,s)} \right),
\]

### Table 5. Results of the convergence study for a single-layer piezoelectric plate with \( a/h = 10 \) in the case of convective boundary conditions

<table>
<thead>
<tr>
<th>( I_1 )</th>
<th>( \bar{\Theta}(-0.5) )</th>
<th>( \bar{\Theta}(0.5) )</th>
<th>( \bar{\varphi}(0.5) )</th>
<th>( \bar{u}_1(0.5) )</th>
<th>( \bar{u}_3(0.5) )</th>
<th>( \bar{\sigma}_{11}(0.5) )</th>
<th>( \bar{\sigma}_{13}(-0.25) )</th>
<th>( \bar{\sigma}_{33}(0) )</th>
<th>( \bar{D}_3(0) )</th>
<th>( \bar{q}_3(0.5) )</th>
<th>( \bar{\eta}(0.5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.72894</td>
<td>0.99045</td>
<td>-10.953</td>
<td>-4.1311</td>
<td>2.8680</td>
<td>-6.9137</td>
<td>1.7532</td>
<td>-9.9104</td>
<td>-3.3167</td>
<td>-2.9865</td>
<td>5.6212</td>
</tr>
<tr>
<td>7</td>
<td>0.72894</td>
<td>0.99045</td>
<td>-10.953</td>
<td>-4.1311</td>
<td>2.8680</td>
<td>-6.7182</td>
<td>1.9635</td>
<td>2.0573</td>
<td>-3.3154</td>
<td>-2.9865</td>
<td>5.6212</td>
</tr>
<tr>
<td>9</td>
<td>0.72894</td>
<td>0.99045</td>
<td>-10.953</td>
<td>-4.1311</td>
<td>2.8680</td>
<td>-6.7182</td>
<td>1.9634</td>
<td>2.0579</td>
<td>-3.3154</td>
<td>-2.9865</td>
<td>5.6212</td>
</tr>
<tr>
<td>11</td>
<td>0.72894</td>
<td>0.99045</td>
<td>-10.953</td>
<td>-4.1311</td>
<td>2.8680</td>
<td>-6.7182</td>
<td>1.9634</td>
<td>2.0579</td>
<td>-3.3154</td>
<td>-2.9865</td>
<td>5.6212</td>
</tr>
<tr>
<td>Dube</td>
<td>0.7289</td>
<td>0.9904</td>
<td>-10.95</td>
<td>-4.131</td>
<td>2.868</td>
<td>-6.718</td>
<td>5.6212</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Fig. 2. Distributions of the temperature, heat flux, electric potential, electric displacement, transverse displacement, transverse stresses, and entropy through the thickness of the single-layer piezoelectric plate for $I_1 = 11$. 
Exact 3D Thermoelectroelastic Analysis

Table 6. Results of the convergence study for a two-layer piezoelectric plate with $a/h = 2$

<table>
<thead>
<tr>
<th>$L_n$</th>
<th>$\tilde{\Theta}(0)$</th>
<th>$\bar{q}(0)$</th>
<th>$\bar{u}_1(0.5)$</th>
<th>$\bar{u}_3(0.5)$</th>
<th>$\bar{\sigma}_{11}(0.5)$</th>
<th>$\bar{\sigma}_{12}(0.5)$</th>
<th>$\bar{\sigma}_{13}(0.5)$</th>
<th>$\bar{\sigma}_{33}(0.5)$</th>
<th>$\bar{D}_3(0)$</th>
<th>$\bar{q}_3(0)$</th>
<th>$\bar{\eta}(0.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.60583</td>
<td>3.3689</td>
<td>-2.3801</td>
<td>10.215</td>
<td>-4.6437</td>
<td>-5.0506</td>
<td>1.3421</td>
<td>-1.7144</td>
<td>-0.17554</td>
<td>3.3443</td>
<td>6.2428</td>
</tr>
<tr>
<td>7</td>
<td>0.60583</td>
<td>3.3689</td>
<td>-2.3801</td>
<td>10.215</td>
<td>-4.6431</td>
<td>-5.0506</td>
<td>1.3549</td>
<td>-1.6464</td>
<td>-0.31134</td>
<td>3.3422</td>
<td>6.2428</td>
</tr>
<tr>
<td>9</td>
<td>0.60583</td>
<td>3.3689</td>
<td>-2.3801</td>
<td>10.215</td>
<td>-4.6432</td>
<td>-5.0506</td>
<td>1.3501</td>
<td>-1.6477</td>
<td>-0.31034</td>
<td>3.3465</td>
<td>6.2428</td>
</tr>
<tr>
<td>11</td>
<td>0.60583</td>
<td>3.3689</td>
<td>-2.3801</td>
<td>10.215</td>
<td>-4.6432</td>
<td>-5.0506</td>
<td>1.3501</td>
<td>-1.6477</td>
<td>-0.31035</td>
<td>3.3465</td>
<td>6.2428</td>
</tr>
</tbody>
</table>

\[ \Pi = \sum_{r,s} \Pi_{rs}(\mu^{(\nu)j_s}_{rs}, \varphi^{(\nu)j_s}_{rs}, \Theta^{(\nu)j_s}_{rs}). \] (73)

Invoking variational equations (31) and (38), we arrive at two systems of linear algebraic equations:

\[ \frac{\partial J_{rs}}{\partial \Theta^{(\nu)j_s}_{rs}} = 0, \] (74)
\[ \frac{\partial \Pi_{rs}}{\partial \mu^{(\nu)j_s}_{rs}} = 0, \frac{\partial \Pi_{rs}}{\partial \varphi^{(\nu)j_s}_{rs}} = 0 \] (75)

of orders $K$ and $4K$, respectively, where $K = \sum_1^N L_n = N + 1$. The linear system (74) is solved first by a method of Gaussian elimination. Next, the linear system (75) is solved using the same method.

The described algorithm was performed with the Symbolic Math Toolbox, which incorporates symbolic computations into the numeric environment of MATLAB. This gives an opportunity to obtain the 3D exact solutions of thermoelasticity for piezoelectric laminated rectangular plates with a specified accuracy.

As a numerical example, we consider a two-layer square plate subjected to the sinusoidally distributed temperature loading on the top surface, whereas the bottom surface is maintained at the reference temperature. The plate with equal thicknesses $h_1 = h_2 = h/2$ is composed of PZT-5A (lower layer) and cadmium selenide (upper layer) polarized in the thickness direction. It is assumed that both outer surfaces are electroded and grounded. Thus, the boundary conditions on the top and bottom surfaces can be written as:

\[ \Theta^{(2)} = \Theta_0 \sin \frac{\pi x_3}{a}, \varphi^{(2)} = \varphi^{(2)} = \sigma^{(2)}_x = \sigma^{(2)}_z = 0 \text{ at } x_3 = \pm h/2, \]
\[ \Theta^{(1)} = \varphi^{(1)} = \sigma^{(1)}_x = \sigma^{(1)}_z = \sigma^{(1)}_3 = 0 \text{ at } x_3 = \mp h/2, \] (76)

where $a = b = 1\text{m}$, $\Theta_0 = 1\text{K}$, and $T_0 = 293\text{K}$. The material properties of PZT-5A and cadmium selenide are presented in [8, 15, 36] and Table 1.

Further, it is convenient to introduce dimensionless variables at crucial points as follows:

\[ \tilde{\Theta} = \Theta(a/2, a/2, z)/\Theta_0, \]
\[ \tilde{\varphi} = 10^3 \varphi(a/2, a/2, z)/E_0\Theta_0, \]
\[ \tilde{\sigma}_{11} = 10^3 \sigma_{11}(a/2, a/2, z)/E_0\Theta_0, \]
\[ \tilde{\sigma}_{12} = 10^3 \sigma_{12}(0, 0, z)/E_0\Theta_0, \]
\[ \tilde{\sigma}_{13} = 10^3 \sigma_{13}(a/2, a/2, z)/E_0\Theta_0, \]
\[ \tilde{\sigma}_{33} = 10^3 \sigma_{33}(a/2, a/2, z)/E_0\Theta_0, \]
\[ \tilde{\eta} = 10^{-3} \eta(a/2, a/2, z)/E_0\Theta_0, S = a/h, z = x_3/h. \] (77)

The reference material data $\sigma_0, k_0, d_0$, and $E_0$ are given in a previous section.

Table 7. Results of the convergence study for a two-layer piezoelectric plate with $a/h = 10$

<table>
<thead>
<tr>
<th>$L_n$</th>
<th>$\tilde{\Theta}(0)$</th>
<th>$\bar{q}(0)$</th>
<th>$\bar{u}_1(0.5)$</th>
<th>$\bar{u}_3(0.5)$</th>
<th>$\bar{\sigma}_{11}(0.5)$</th>
<th>$\bar{\sigma}_{12}(0.5)$</th>
<th>$\bar{\sigma}_{13}(0.5)$</th>
<th>$\bar{\sigma}_{33}(0.5)$</th>
<th>$\bar{D}_3(0)$</th>
<th>$\bar{q}_3(0)$</th>
<th>$\bar{\eta}(0.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.86748</td>
<td>6.6726</td>
<td>-2.8455</td>
<td>6.2730</td>
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<td>-6.0384</td>
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<td>-6.3918</td>
<td>1.8001</td>
<td>-3.5268</td>
<td>6.2434</td>
</tr>
<tr>
<td>7</td>
<td>0.86748</td>
<td>6.6726</td>
<td>-2.8455</td>
<td>6.2730</td>
<td>-1.6989</td>
<td>-6.0384</td>
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<td>-6.3860</td>
<td>1.8020</td>
<td>-3.5268</td>
<td>6.2434</td>
</tr>
<tr>
<td>9</td>
<td>0.86748</td>
<td>6.6726</td>
<td>-2.8455</td>
<td>6.2730</td>
<td>-1.6989</td>
<td>-6.0384</td>
<td>2.7132</td>
<td>-6.3876</td>
<td>1.8020</td>
<td>-3.5268</td>
<td>6.2434</td>
</tr>
</tbody>
</table>
Tables 6 and 7 show again the high potential of the SaS method, which yields the exact solution of 3D thermoelectroelasticity for piezoelectric laminated rectangular plates with a prescribed accuracy by using the sufficiently large number of SaS inside layers $I_1$ and $I_2$. Figure 3 presents through-the-thickness distributions of the temperature, heat flux, electric potential, electric displacement, displacements, stresses, and entropy for different slenderness ratios $a/h$ employing nine SaS for each layer. It is seen that the boundary conditions for transverse stresses on the bottom and top surfaces and the continuity conditions for transverse components of the heat flux, electric displacement, and stress tensor at the layer interface are satisfied exactly. As we remember, these functions can be easily found via constitutive equations (35), (48), and (49).

9. Conclusions

An efficient method of solving the steady-state problem of 3D thermoelectroelasticity for piezoelectric laminated plates has been proposed. It is based on the new method of SaS located at Chebyshev polynomial nodes inside the layers and interfaces as well. The analysis of piezoelectric plates is based on the 3D constitutive equations and gives an opportunity to obtain exact 3D solutions of piezoelectricity for thick and thin plates with a specified accuracy.

Funding

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References


Fig. 3. Distributions of the temperature, heat flux, electric potential, electric displacement, displacements, stresses, and entropy through the thickness of the two-layer piezoelectric plate for $I_1 = I_2 = 9$. 


