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Equivalent Single-Layer and Layerwise Shell Theories and Rigid-Body Motions—Part I: Foundations

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The exact representation of rigid-body motions in the displacement patterns of the first-order equivalent single-layer (ESL) and layerwise (LW) shell elements is considered. This consideration requires the development of the strain–displacement relationships of the ESL and LW shell theories with regard to their consistency with rigid-body motions. The fundamental unknowns consist of six displacements of the face surfaces of the shell in the ESL theory and 3(N+1) displacements of the face surfaces of layers in the LW shell theory, where N is a number of layers. Such a choice of displacements makes it possible to deduce strain–displacement relationships, which are objective, that is, invariant under rigid-body motions. To overcome thickness locking, three types of the modified material stiffness matrix corresponding to the generalized plane-stress state are employed.

§1. INTRODUCTION

One of the main requirements of a finite element that is intended for the general analysis of shells is that it must lead to strain-free modes for rigid-body motions. The adequate representation of rigid-body motions is a necessary condition if an element is to have good accuracy and convergence properties. Therefore, when an inconsistent shell theory is used to construct any finite element, erroneous straining modes under rigid-body motions may appear. This problem has only been studied for the Kirchhoff–Love shell theory [1–3] and Timoshenko–Mindlin-type shell theory [4, 5]. Herein, first, the more general study on the basis of the first-order equivalent single-layer (ESL) shell theory taking into account the transverse normal deformation response is considered. Six displacements of the face surfaces of the shell are selected as unknown functions. Second, an approach on the basis of the first-order layerwise (LW) shell theory allowing for thickness stretching is also presented. Note that a comprehensive discussion on the use of the LW shell models in engineering applications may be found in books [6, 7] and overview works [8–11]. In our case, the fundamental unknowns consist of 3(N + 1) displacements of the face surfaces of layers [12], where N is a number of layers. Such a choice of unknowns allowed us to deduce strain–displacement relationships of the ESL and LW shell theories, which are completely free for all rigid-body motions.

It should be observed that a close six-parameter geometrically exact shell model was proposed by Simo et al. [13] where covariant derivatives, associated with the Riemannian connection on the reference surface, do not explicitly appear in the formulation. The feature of our geometrically exact shell model is that, in contrast with the aforementioned work, coefficients of the second fundamental form appear in the formulation. This clears the way to elaborate the more robust numerical algorithms because during the geometrical modeling in CAD systems the surfaces are usually generated by nonuniform rational B-spline (NURBS) functions [14]. So, the exact NURBS shell surface functions may be directly used for describing the reference surface that would yield an efficient numerical implementation which will be free from mathematical complexities and suitable for large-scale computations.

It is common knowledge that six-parameter ESL shell or 3(N + 1)-parameter LW shell formulations based on the complete three-dimensional (3D) constitutive equations are deficient because so-called thickness locking [15, 16] can occur. This phenomenon occurs in bending-dominated shell problems when Poisson ratios are not equal to zero. To avoid thickness locking at the *finite element level*, an efficient enhanced assumed strain method [15, 17] may be applied. To circumvent locking phenomena at both the mechanical and computational levels, the 3D constitutive equations have to be modified. For this purpose, three simple and effective remedies may be employed, namely, the ad hoc modified laminate stiffness matrix [18–20] and simplified material stiffness matrices, symmetric [5, 21–23], or nonsymmetric [12, 24, 25], corresponding to the generalized plane-stress condition. Herein, all remedies are introduced into the formulation by using the unified technique that allows one to assess their advantages and disadvantages.

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Taking into account that displacement vectors of bottom and top surfaces of the shell or face surfaces of layers are represented in the *reference surface* basis, the proposed formulations have substantial computational advantages compared to the conventional isoparametric finite element formulations, because they reduce the costly numerical integration by deriving the stiffness matrices. Besides, element matrices require only direct substitutions; that is, no inversion is needed if sides of the element coincide with lines of principal curvatures of the reference surface and they are evaluated by using the 3D *analytical* integration. This will be discussed in a companion paper [26].

§2. PROBLEM FORMULATION

2.1. Preliminaries

Consider a shell built up in the general case by the arbitrary superposition across the wall thickness of N layers of uniform thickness h_k. The kth layer may be defined as a 3D body of volume V_k bounded by two surfaces, S_{k-1} and S_k, located at the distances δ_{k-1} and δ_k measured with respect to the reference surface S, and the edge boundary surface Ω_k (Figure 1). The full edge boundary surface $\Omega = \Omega_1 + \Omega_2 + \cdots + \Omega_N$ is generated by the normals to the reference surface along the bounding curve $\Gamma \subset S$. It is also assumed that the bounding surfaces S_{k-1} and S_k are continuous, sufficiently smooth, and without any singularities. Let the reference surface S be referred to as the orthogonal curvilinear coordinate system α_1 and α_2, which coincides with the lines of principal curvatures of its surface, whereas coordinate α_3 is oriented along the unit vector $\boldsymbol{a}_3 = \boldsymbol{e}_3$ normal to the reference surface; $\boldsymbol{a}_\alpha = A_\alpha \boldsymbol{e}_\alpha$ are the basis vectors of the reference surface S; $\boldsymbol{g}_\alpha^{(n)} = A_\alpha^{(n)} \boldsymbol{e}_\alpha$ are the basis vectors of the face surfaces of the kth layer; $\boldsymbol{g}_\alpha^- = \boldsymbol{g}_\alpha^{(0)}$ and $\boldsymbol{g}_\alpha^+ = \boldsymbol{g}_\alpha^{(N)}$ are the basis vectors of the bottom and top surfaces of the shell; $\boldsymbol{e}_\alpha$ are the tangent unit vectors to the lines of principal curvatures of the reference surface; A_α and $A_\alpha^{(n)}$ are the Lamé coefficients of the reference surface and face surfaces of the kth layer. Here and in the following developments, the index k identifies the belonging of any quantity to the kth layer and runs from 1 to N; the index n identifies the belonging of any quantity to the face surfaces of the kth layer and runs from $k-1$ to k; the abbreviation $()_{,\alpha}$ implies the partial derivatives with respect to coordinates α_1 and α_2; indices i, j, ℓ, and m take the values 1, 2, and 3, while Greek indices α, β, γ, and δ take the values 1 and 2.

The constituent layers of the shell are supposed to be rigidly joined so that no slip on contact surfaces and no separation of layers can occur. The material of each constituent layer is assumed to be linearly elastic, anisotropic, homogeneous, or fiber reinforced such that in each point there is a single surface of elastic symmetry parallel to the reference surface. Let p_i^- and p_i^+ be the components of the external loading vectors $\boldsymbol{p}^-$ and $\boldsymbol{p}^+$ acting on

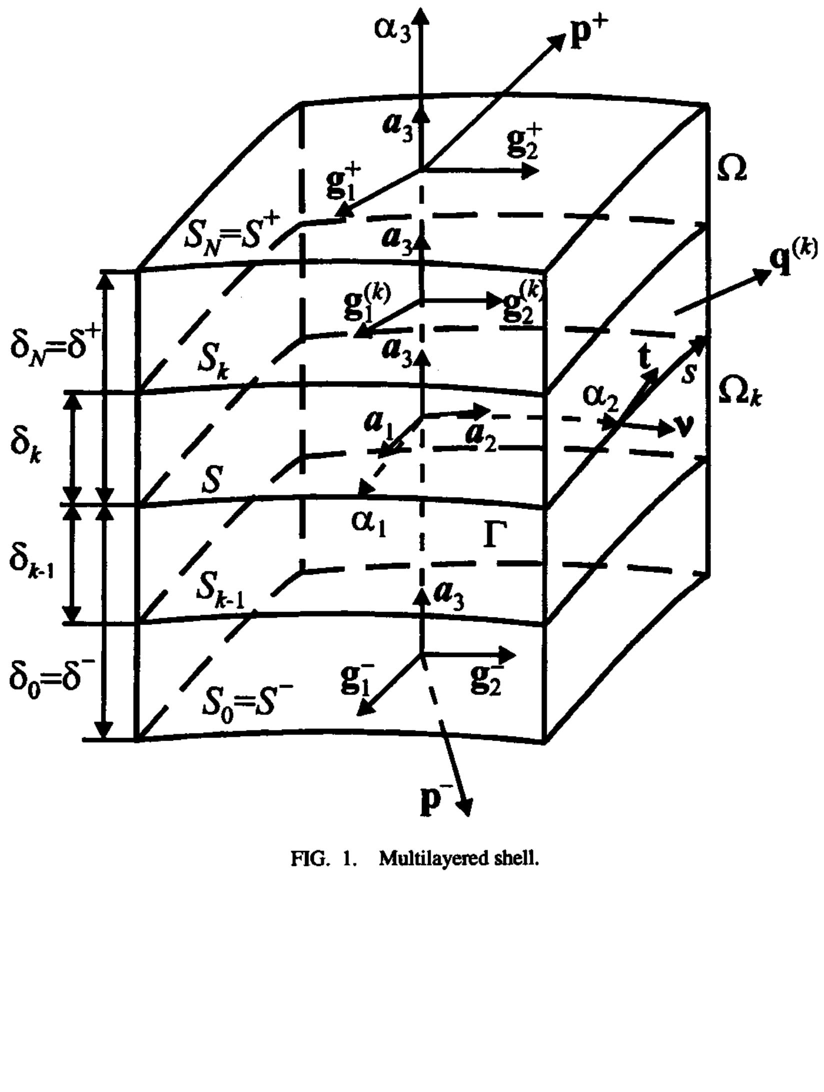

FIG. 1. Multilayered shell.

the bottom surface $S^- = S_0$ and top surface $S^+ = S_N$ in the α_i coordinate directions, respectively; $\boldsymbol{q}^{(k)} = q_v^{(k)}\boldsymbol{v} + q_t^{(k)}\boldsymbol{t} + q_3^{(k)}\boldsymbol{e}_3$ is the external loading vector acting on the edge boundary surface Ω_k, where $q_v^{(k)}$, $q_t^{(k)}$, and $q_3^{(k)}$ are the components of its vector in the v, t, and α_3 directions, respectively; $\boldsymbol{v}$ and $\mathbf{t}$ are the normal and tangential unit vectors to the bounding curve Γ.

2.2. ESL Shell Kinematics

The ESL shell theory is based on the linear approximation of displacements in the thickness direction:

$$\hat{\boldsymbol{R}} = N^-\hat{\boldsymbol{R}}^- + N^+\hat{\boldsymbol{R}}^+ \tag{1a}$$

$$\hat{\boldsymbol{R}}^\pm = \boldsymbol{R}^\pm + \boldsymbol{v}^\pm \qquad \boldsymbol{R}^\pm = \boldsymbol{r} + \delta^\pm \boldsymbol{e}_3$$

$$N^- = \frac{1}{h}(\delta^+ - \alpha_3) \qquad N^+ = \frac{1}{h}(\alpha_3 - \delta^-) \tag{1b}$$

$$\boldsymbol{v}^\pm = \sum_i v_i^\pm \boldsymbol{e}_i \tag{1c}$$

where $\boldsymbol{r}(\alpha_1, \alpha_2)$ is the position vector of the reference surface; $\boldsymbol{R}^\pm$ and $\hat{\boldsymbol{R}}^\pm$ are the position vectors of the face surfaces in the initial and final shell configurations (Figure 2); $\boldsymbol{v}^\pm$ are the displacement vectors of the face surfaces; $v_i^\pm(\alpha_1, \alpha_2)$ are the components of these vectors; $N^\pm(\alpha_3)$ are the linear through-the-thickness shape functions; and h is the thickness of the shell. It is important that displacement vectors (1c) are represented in the orthonormal reference surface basis $\boldsymbol{e}_i$, which allows one to reduce the costly numerical integration by deriving the elemental stiffness matrix.

The strain–displacement relationships of the first-order ESL shell theory [5] can be written in a more convenient form for the finite element implementation as

$$\varepsilon_{\alpha\beta} = N^- e_{\alpha\beta}^- + N^+ e_{\alpha\beta}^+ \tag{2a}$$

$$\varepsilon_{\alpha_3} = N^- e_{\alpha_3}^- + N^+ e_{\alpha 3}^+ \qquad \varepsilon_{33} = e_{33} \tag{2b}$$

where $e_{\alpha\beta}^\pm$ and $e_{\alpha 3}^\pm$ are the tangential and transverse shear components of the strain tensor of face surfaces $S^\pm$, defined by

$$2e_{\alpha\beta}^\pm = \frac{1}{A_\alpha A_\beta}(\hat{\boldsymbol{g}}_\alpha^\pm \cdot \hat{\boldsymbol{g}}_\beta^\pm - \boldsymbol{g}_\alpha^\pm \cdot \boldsymbol{g}_\beta^\pm)$$

$$= \frac{\zeta_\beta^\pm}{A_\alpha}\boldsymbol{v}_{,\alpha}^\pm \cdot \boldsymbol{e}_\beta + \frac{\zeta_\alpha^\pm}{A_\beta}\boldsymbol{v}_{,\beta}^\pm \cdot \boldsymbol{e}_\alpha \tag{3a}$$

$$2e_{\alpha_3}^\pm = \frac{1}{A_\alpha}(\hat{\boldsymbol{g}}_\alpha^\pm \cdot \hat{\boldsymbol{a}}_3 - \boldsymbol{g}_\alpha^\pm \cdot \boldsymbol{a}_3)$$

$$= \zeta_\alpha^\pm \boldsymbol{\beta} \cdot \boldsymbol{e}_\alpha + \frac{1}{A_\alpha}\boldsymbol{v}_{,\alpha}^\pm \cdot \boldsymbol{e}_3 \tag{3b}$$

$$2e_{33} = \hat{\boldsymbol{a}}_3 \cdot \hat{\boldsymbol{a}}_3 - \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = 2\boldsymbol{\beta} \cdot \boldsymbol{e}_3$$

$$\hat{\boldsymbol{g}}_\alpha^\pm = \hat{\boldsymbol{R}}_{,\alpha}^\pm = \boldsymbol{g}_\alpha^\pm + \boldsymbol{v}_{,\alpha}^\pm \qquad \boldsymbol{g}_\alpha^\pm = \boldsymbol{R}_{,\alpha}^\pm = A_\alpha^\pm \boldsymbol{e}_\alpha$$

$$\hat{\boldsymbol{a}}_3 = \boldsymbol{a}_3 + \boldsymbol{\beta} \tag{3c}$$

$$\boldsymbol{\beta} = \frac{1}{h}(\boldsymbol{v}^+ - \boldsymbol{v}^-) \qquad A_2^\pm = A_\alpha \zeta_\alpha^\pm$$

$$\zeta_\alpha^\pm = 1 + k_\alpha \delta^\pm \tag{3d}$$

where k_α are the principal curvatures of the reference surface. Note that in Eqs. (3a) and (3b) only the linear terms are retained. Strain–displacement relationships (2) and (3) are very attractive because they are objective (i.e., invariant under rigid-body motions), which will be discussed in Section 2.4.

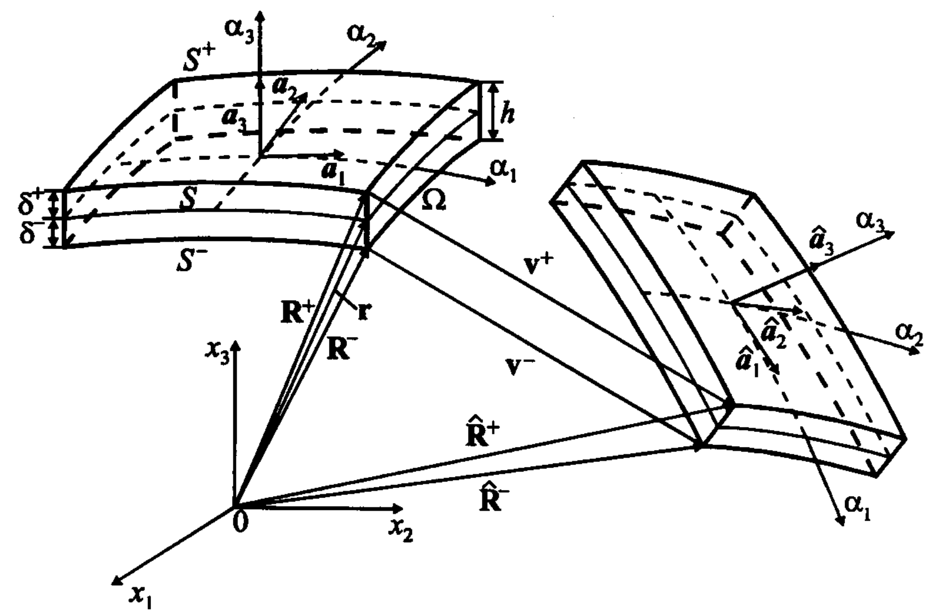

FIG. 2. Geometry and kinematics of shell.

We further prove a fundamental result concerning the relation between transverse strain components (3*b*).

Proposition 1 *The transverse components of the developed strain tensor satisfy the following coupling conditions*:

$$2(e_{\alpha3}^{+} - e_{\alpha3}^{-}) = \frac{1}{A_\alpha} h e_{33,\alpha} \tag{4}$$

Proof. Using Eqs. (3*b*) and (3*c*) yields

$$2(e_{\alpha3}^{+} - e_{\alpha3}^{-}) = \frac{1}{A_\alpha} h(A_\alpha k_\alpha \boldsymbol{\beta}\cdot\boldsymbol{e}_\alpha + \boldsymbol{\beta}_{,\alpha}\cdot\boldsymbol{e}_3) \tag{5}$$

Taking into account a formula for derivatives of the basis vector $\boldsymbol{e}_3$ with respect to orthogonal curvilinear coordinates α_1 and α_2 [1],

$$\frac{1}{A_\alpha}\boldsymbol{e}_{3,\alpha} = k_\alpha \boldsymbol{e}_\alpha \tag{6}$$

one derives the following from Eq. (3*b*):

$$e_{33,\alpha} = \boldsymbol{\beta}\cdot\boldsymbol{e}_{3,\alpha} + \boldsymbol{\beta}_{,\alpha}\cdot\boldsymbol{e}_3 = A_\alpha k_\alpha \boldsymbol{\beta}\cdot\boldsymbol{e}_\alpha + \boldsymbol{\beta}_{,\alpha}\cdot\boldsymbol{e}_3 \tag{7}$$

Required relation (4) immediately follows from Eqs. (5) and (7). □

It should be mentioned that coupling conditions (4) play a central role in our finite element formulation, as we will see in the companion to this paper [26].

Substituting displacements (1*c*) into the strain–displacement equations (3) and again allowing for Eq. (6) and expressions for derivatives of the basis vectors $\boldsymbol{e}_\alpha$ along coordinate lines [1],

$$\frac{1}{A_\alpha}\boldsymbol{e}_{\alpha,\alpha} = -B_{\alpha\beta}\boldsymbol{e}_\beta - k_\alpha\boldsymbol{e}_3 \qquad \frac{1}{A_\alpha}\boldsymbol{e}_{\beta,\alpha} = B_{2\beta,\beta}\boldsymbol{e}_\alpha \qquad \beta\neq\alpha \tag{8}$$

one can write these equations in a scalar form:

$$e_{\alpha\alpha}^{\pm} = \zeta_\alpha^{\pm}\lambda_\alpha^{\pm} \qquad 2e_{12}^{\pm} = \zeta_2^{\pm}\omega_1^{\pm} + \zeta_1^{\pm}\omega_2^{\pm} \tag{9a}$$

$$2e_{\alpha3}^{\pm} = \zeta_\alpha^{\pm}\beta_\alpha - \theta_\alpha^{\pm} \qquad e_{33} = \beta_3 \tag{9b}$$

where

$$\begin{aligned}
\lambda_\alpha^{\pm} &= \left(\frac{1}{A_\alpha}v_\alpha^{\pm}\right)_{,\alpha} + B_{\alpha\alpha}v_\alpha^{\pm} + B_{\alpha\beta}v_\beta^{\pm} + k_\alpha v_3^{\pm} \\
&\qquad \beta\neq\alpha \\
\omega_\alpha^{\pm} &= \left(\frac{1}{A_\alpha}v_\beta^{\pm}\right)_{,\alpha} + B_{\alpha\alpha}v_\beta^{\pm} - B_{\alpha\beta}v_\alpha^{\pm} \qquad \beta\neq\alpha \\
\theta_\alpha^{\pm} &= -\left(\frac{1}{A_\alpha}v_3^{\pm}\right)_{,\alpha} - B_{\alpha\alpha}v_3^{\pm} + k_\alpha v_\alpha^{\pm} \\
\beta_i &= \frac{1}{h}(v_i^{+} - v_i^{-}) \qquad B_{\gamma\delta} = \frac{1}{A_\gamma A_\delta}A_{\gamma,\delta}
\end{aligned} \tag{10}$$

It should be mentioned that derivatives in Eq. (10) have been written in a form that is best suited for applying high-performance analytical integration schemes inside the element.

2.3. LW Shell Kinematics

The LW shell theory is based on the linear approximation of displacements in the thickness direction of the kth layer:

$$\hat{\boldsymbol{R}} = N_k^{-}\hat{\boldsymbol{R}}^{(k-1)} + N_k^{+}\hat{\boldsymbol{R}}^{(k)} \tag{11a}$$

$$\hat{\boldsymbol{R}}^{(n)} = \boldsymbol{R}^{(n)} + \boldsymbol{v}^{(n)} \qquad \boldsymbol{R}^{(n)} = \boldsymbol{r} + \delta_n\boldsymbol{e}_3 \tag{11b}$$

$$N_k^{-} = \frac{1}{h_k}(\delta_k - \alpha_3) \qquad N_k^{+} = \frac{1}{h_k}(\alpha_3 - \delta_{k-1})$$

$$\boldsymbol{v}^{(n)} = \sum_i v_i^{(n)}\boldsymbol{e}_i \tag{11c}$$

where $\boldsymbol{R}^{(n)}$ and $\hat{\boldsymbol{R}}^{(n)}$ are the position vectors of the face surfaces of the kth layer in initial and final shell configurations; $\boldsymbol{v}^{(n)}$ are the displacement vectors of the face surfaces of the kth layer; $v_i^{(n)}(\alpha_1, \alpha_2)$ are the components of these vectors; $N_k^{\pm}(\alpha_3)$ are the linear through-the-thickness shape functions of the kth layer; and, as we remember, the index n runs from $k-1$ to k.

The strain–displacement relationships of the first-order LW shell theory can be represented by

$$\varepsilon_{\alpha\beta}^{(k)} = N_k^{-}e_{\alpha\beta}^{(k-1)} + N_k^{+}e_{\alpha\beta}^{(k)} \tag{12a}$$

$$\varepsilon_{\alpha3}^{(k)} = N_k^{-}e_{\alpha3}^{(k)-} + N_k^{+}e_{\alpha3}^{(k)+} \qquad \varepsilon_{33}^{(k)} = e_{33}^{(k)} \tag{12b}$$

where $e_{\alpha\beta}^{(k-1)}, e_{\alpha\beta}^{(k)}$ and $e_{\alpha3}^{(k)-}, e_{\alpha3}^{(k)+}$ are the tangential and transverse shear strain components of the face surfaces of the kth layer, defined as

$$\begin{aligned}
2e_{\alpha\beta}^{(n)} &= \frac{1}{A_\alpha A_\beta}\left(\hat{\boldsymbol{g}}_\alpha^{(n)}\cdot\hat{\boldsymbol{g}}_\beta^{(n)} - \boldsymbol{g}_\alpha^{(n)}\cdot\boldsymbol{g}_\beta^{(n)}\right) \\
&= \frac{\zeta_\beta^{(n)}}{A_\alpha}\boldsymbol{v}_{,\alpha}^{(n)}\cdot\boldsymbol{e}_\beta + \frac{\zeta_\alpha^{(n)}}{A_\beta}\boldsymbol{v}_{,\beta}^{(n)}\cdot\boldsymbol{e}_\alpha
\end{aligned} \tag{13a}$$

$$\begin{aligned}
2e_{\alpha3}^{(k)-} &= \frac{1}{A_\alpha}\left(\hat{\boldsymbol{g}}_\alpha^{(k-1)}\cdot\hat{\boldsymbol{g}}_3^{(k)} - \boldsymbol{g}_\alpha^{(k-1)}\cdot\boldsymbol{g}_3^{(k)}\right) \\
&= \zeta_\alpha^{(k-1)}\boldsymbol{\beta}^{(k)}\cdot\boldsymbol{e}_\alpha + \frac{1}{A_\alpha}\boldsymbol{v}_{,\alpha}^{(k-1)}\cdot\boldsymbol{e}_3
\end{aligned} \tag{13b}$$

$$\begin{aligned}
2e_{\alpha3}^{(k)+} &= \frac{1}{A_\alpha}\left(\hat{\boldsymbol{g}}_\alpha^{(k)}\cdot\hat{\boldsymbol{g}}_3^{(k)} - \boldsymbol{g}_\alpha^{(k)}\cdot\boldsymbol{g}_3^{(k)}\right) = \zeta_\alpha^{(k)}\boldsymbol{\beta}^{(k)}\cdot\boldsymbol{e}_\alpha + \frac{1}{A_\alpha}\boldsymbol{v}_{,\alpha}^{(k)}\cdot\boldsymbol{e}_3 \\
2e_{33}^{(k)} &= \hat{\boldsymbol{g}}_3^{(k)}\cdot\hat{\boldsymbol{g}}_3^{(k)} - \boldsymbol{g}_3^{(k)}\cdot\boldsymbol{g}_3^{(k)} = 2\boldsymbol{\beta}^{(k)}\cdot\boldsymbol{e}_3 \\
\hat{\boldsymbol{g}}_\alpha^{(n)} &= \hat{\boldsymbol{R}}_{,\alpha}^{(n)} = \boldsymbol{g}_\alpha^{(n)} + \boldsymbol{v}_{,\alpha}^{(n)} \qquad \boldsymbol{g}_\alpha^{(n)} = \boldsymbol{R}_{,\alpha}^{(n)} = A_\alpha^{(n)}\boldsymbol{e}_\alpha \\
\hat{\boldsymbol{g}}_3^{(k)} &= \boldsymbol{e}_3 + \boldsymbol{\beta}^{(k)} \qquad \boldsymbol{g}_3^{(k)} = \boldsymbol{e}_3 \\
\boldsymbol{\beta}^{(k)} &= \frac{1}{h_k}\left(\boldsymbol{v}^{(k)} - \boldsymbol{v}^{(k-1)}\right) \qquad A_\alpha^{(n)} = A_\alpha\zeta_\alpha^{(n)} \\
\zeta_\alpha^{(n)} &= 1 + k_\alpha\delta_n
\end{aligned} \tag{13c}$$

Note that strain–displacement relationships (12) and (13) are also objective. A proof of this statement is given in the next section.

Proposition 2 *The transverse components of the developed strain tensor for the LW shell theory satisfy the following coupling conditions*:

$$2\left(e_{\alpha 3}^{(k)+} - e_{\alpha 3}^{(k)-}\right) = \frac{1}{A_\alpha} h_k e_{33,\alpha}^{(k)} \tag{14}$$

Proof. Needed relations (14) may be deduced by using a technique developed in Proposition 1. □

Finally, taking into account formulas for the derivatives, Eqs. (6) and (8), we represent strain–displacement relationships (13) in a scalar form:

$$e_{\alpha\alpha}^{(n)} = \zeta_\alpha^{(n)} \lambda_\alpha^{(n)} \qquad 2e_{12}^{(n)} = \zeta_2^{(n)} \omega_1^{(n)} + \zeta_1^{(n)} \omega_2^{(n)} \tag{15a}$$

$$2e_{\alpha 3}^{(k)-} = \zeta_\alpha^{(k-1)} \beta_\alpha^{(k)} - \theta_\alpha^{(k-1)} \qquad 2e_{\alpha 3}^{(k)+} = \zeta_\alpha^{(k)} \beta_\alpha^{(k)} - \theta_\alpha^{(k)}$$

$$e_{33}^{(k)} = \beta_3^{(k)} \tag{15b}$$

where

$$\lambda_\alpha^{(n)} = \left(\frac{1}{A_\alpha} v_\alpha^{(n)}\right)_{,\alpha} + B_{\alpha\alpha} v_\alpha^{(n)} + B_{\alpha\beta} v_\beta^{(n)} + k_\alpha v_3^{(n)}$$

$$\beta \neq \alpha \tag{16}$$

$$\omega_\alpha^{(n)} = \left(\frac{1}{A_\alpha} v_\beta^{(n)}\right)_{,\alpha} + B_{\alpha\alpha} v_\beta^{(n)} - B_{\alpha\beta} v_\alpha^{(n)} \qquad \beta \neq \alpha$$

$$\theta_\alpha^{(n)} = -\left(\frac{1}{A_\alpha} v_3^{(n)}\right)_{,\alpha} - B_{\alpha\alpha} v_3^{(n)} + k_\alpha v_\alpha^{(n)}$$

$$\beta_i^{(k)} = \frac{1}{h_k}\left(v_i^{(k)} - v_i^{(k-1)}\right)$$

The geometrical parameters of the reference surface $B_{\alpha\beta}$ are given, as we remember, in Eq. (10).

2.4. Rigid-Body Motions

A small rigid-body motion is defined as [1]

$$\boldsymbol{u}^R = \boldsymbol{\Delta} + \boldsymbol{\Phi} \times \boldsymbol{R} \tag{17}$$

$$\boldsymbol{R} = \boldsymbol{r} + \alpha_3 \boldsymbol{e}_3 \qquad \boldsymbol{\Delta} = \sum_i \Delta_i \boldsymbol{e}_i \qquad \boldsymbol{\Phi} = \sum_i \Phi_i \boldsymbol{e}_i$$

where $\boldsymbol{R}$ is the position vector of any point of the shell, $\boldsymbol{\Delta}$ is the constant displacement (translation) vector, and $\boldsymbol{\Phi}$ is the constant rotation vector.

In particular, rigid-body motions of the face surfaces of the shell will be given by

$$\boldsymbol{v}^{\pm R} = \boldsymbol{\Delta} + \boldsymbol{\Phi} \times \boldsymbol{R}^{\pm} \tag{18}$$

The derivatives of the translation and rotation vectors with respect to the curvilinear reference surface coordinates α_1 and α_2 are zero; that is,

$$\boldsymbol{\Delta}_{,\alpha} = \boldsymbol{0} \qquad \boldsymbol{\Phi}_{,\alpha} = \boldsymbol{0} \tag{19}$$

By using Eqs. (1*b*), (6), (18), and (19), one can obtain an expression for derivatives:

$$\boldsymbol{v}_{,\alpha}^{\pm R} = A_\alpha^{\pm} \boldsymbol{\Phi} \times \boldsymbol{e}_\alpha \tag{20}$$

It may be verified by means of Eqs. (18) and (20) that strains given by relationships (2) and (3) are all zero in a rigid-body motion, because

$$2e_{\alpha\beta}^{\pm R} = \zeta_\alpha^{\pm} \zeta_\beta^{\pm} [(\boldsymbol{\Phi} \times \boldsymbol{e}_\alpha) \cdot \boldsymbol{e}_\beta + (\boldsymbol{\Phi} \times \boldsymbol{e}_\beta) \cdot \boldsymbol{e}_\alpha] = 0 \tag{21a}$$

$$2e_{\alpha 3}^{\pm R} = \zeta_\alpha^{\pm} [(\boldsymbol{\Phi} \times \boldsymbol{e}_3) \cdot \boldsymbol{e}_\alpha + (\boldsymbol{\Phi} \times \boldsymbol{e}_\alpha) \cdot \boldsymbol{e}_3] = 0$$

$$e_{33}^{R} = (\boldsymbol{\Phi} \times \boldsymbol{e}_3) \cdot \boldsymbol{e}_3 = 0 \tag{21b}$$

Therefore strain–displacement relationships of the ESL shell theory are invariant under rigid-body motions.

Furthermore, we represent rigid-body motions of the face surfaces of the kth layer as

$$\boldsymbol{v}^{(n)R} = \boldsymbol{\Delta} + \boldsymbol{\Phi} \times \boldsymbol{R}^{(n)} \tag{22}$$

Again using Eqs. (6) and (19) together with Eqs. (11*b*) and (22), one derives the following formula for derivatives:

$$\boldsymbol{v}_{,\alpha}^{(n)R} = A_\alpha^{(n)} \boldsymbol{\Phi} \times \boldsymbol{e}_\alpha \tag{23}$$

Allowing for Eqs. (22) and (23), we can prove a fundamental statement that strain–displacement relationships of the LW shell theory (12) and (13) are also invariant under rigid-body motions:

$$2e_{\alpha\beta}^{(n)R} = \zeta_\alpha^{(n)} \zeta_\beta^{(n)} [(\boldsymbol{\Phi} \times \boldsymbol{e}_\alpha) \cdot \boldsymbol{e}_\beta + (\boldsymbol{\Phi} \times \boldsymbol{e}_\beta) \cdot \boldsymbol{e}_\alpha] = 0 \tag{24a}$$

$$2e_{\alpha 3}^{(k)-R} = \zeta_\alpha^{(k-1)} [(\boldsymbol{\Phi} \times \boldsymbol{e}_3) \cdot \boldsymbol{e}_\alpha + (\boldsymbol{\Phi} \times \boldsymbol{e}_\alpha) \cdot \boldsymbol{e}_3] = 0 \tag{24b}$$

$$2e_{\alpha 3}^{(k)+R} = \zeta_\alpha^{(k)} [(\boldsymbol{\Phi} \times \boldsymbol{e}_3) \cdot \boldsymbol{e}_\alpha + (\boldsymbol{\Phi} \times \boldsymbol{e}_\alpha) \cdot \boldsymbol{e}_3] = 0$$

$$e_{33}^{(k)R} = (\boldsymbol{\Phi} \times \boldsymbol{e}_3) \cdot \boldsymbol{e}_3 = 0$$

§3. HU–WASHIZU VARIATIONAL EQUATION

It is well known that the Hu–Washizu variational principle provides the basis for derivation of various variational principles, and many different mixed and hybrid finite elements for multilayered shells may be designed. Here and in the following developments, for conciseness, we will consider only one theory, namely, the first-order ESL shell theory, which will demonstrate the feature of our approach.

The ESL shell theory is developed based on the assumed approximations of displacements (1) and displacement-dependent strains (2) in the thickness direction. Additionally, one should adopt the similar approximation for the assumed displacement-independent strains:

$$\varepsilon_{\alpha\beta}^{AS} = N^- E_{\alpha\beta}^- + N^+ E_{\alpha\beta}^+ \tag{25a}$$

$$\varepsilon_{\alpha 3}^{AS} = N^- E_{\alpha 3}^- + N^+ E_{\alpha 3}^+ \qquad \varepsilon_{33}^{AS} = E_{33} \tag{25b}$$

Substituting approximations (1c), (2), and (25) into the Hu–Washizu mixed variational principle [27] and accounting for the fact that the metrics of all surfaces parallel to the reference surface are identical and equal to the metric of the middle surface, one can derive

$$\int_S\int[(\boldsymbol{H}-\boldsymbol{D}\boldsymbol{E})^{\mathrm{T}}\delta\boldsymbol{E}+(\boldsymbol{E}-\boldsymbol{e})^{\mathrm{T}}\delta\boldsymbol{H}-\boldsymbol{H}^{\mathrm{T}}\delta\boldsymbol{e}+\boldsymbol{P}^{\mathrm{T}}\delta\boldsymbol{v}]$$
$$\times\bar{A}_1\bar{A}_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2+\oint_\Gamma\hat{\boldsymbol{H}}_\Gamma^{\mathrm{T}}\delta\boldsymbol{v}_\Gamma(1+k_N\bar{\delta})\mathrm{d}s=0 \quad (26)$$

where matrix notations are introduced:

$$\boldsymbol{D}=\begin{bmatrix}
D^{00}_{1111} & D^{01}_{1111} & D^{00}_{1122} & D^{01}_{1122} & D^{00}_{1112} & D^{01}_{1112} & 0 & 0 & 0 & 0 & D^{0}_{1133}\\
D^{01}_{1111} & D^{11}_{1111} & D^{01}_{1122} & D^{11}_{1122} & D^{01}_{1112} & D^{11}_{1112} & 0 & 0 & 0 & 0 & D^{1}_{1133}\\
D^{00}_{2211} & D^{01}_{2211} & D^{00}_{2222} & D^{01}_{2222} & D^{00}_{2212} & D^{01}_{2212} & 0 & 0 & 0 & 0 & D^{0}_{2233}\\
D^{01}_{2211} & D^{11}_{2211} & D^{01}_{2222} & D^{11}_{2222} & D^{01}_{2212} & D^{11}_{2212} & 0 & 0 & 0 & 0 & D^{1}_{2233}\\
D^{00}_{1211} & D^{01}_{1211} & D^{00}_{1222} & D^{01}_{1222} & D^{00}_{1212} & D^{01}_{1212} & 0 & 0 & 0 & 0 & D^{0}_{1233}\\
D^{01}_{1211} & D^{11}_{1211} & D^{01}_{1222} & D^{11}_{1222} & D^{01}_{1212} & D^{11}_{1212} & 0 & 0 & 0 & 0 & D^{1}_{1233}\\
0 & 0 & 0 & 0 & 0 & 0 & D^{00}_{1313} & D^{01}_{1313} & D^{00}_{1323} & D^{01}_{1323} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & D^{01}_{1313} & D^{11}_{1313} & D^{01}_{1323} & D^{11}_{1323} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & D^{00}_{2313} & D^{01}_{2313} & D^{00}_{2323} & D^{01}_{2323} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & D^{01}_{2313} & D^{11}_{2313} & D^{01}_{2323} & D^{11}_{2323} & 0\\
D^{0}_{3311} & D^{1}_{3311} & D^{0}_{3322} & D^{1}_{3322} & D^{0}_{3312} & D^{1}_{3312} & 0 & 0 & 0 & 0 & D_{3333}
\end{bmatrix} \quad (27)$$

$$\boldsymbol{v}=[v_1^-\ v_1^+\ v_2^-\ v_2^+\ v_3^-\ v_3^+]^{\mathrm{T}} \qquad \boldsymbol{v}_\Gamma=[v_\nu^-\ v_\nu^+\ v_t^-\ v_t^+\ v_3^-\ v_3^+]^{\mathrm{T}}$$
$$\boldsymbol{e}=[e_{11}^-\ e_{11}^+\ e_{22}^-\ e_{22}^+\ 2e_{12}^-\ 2e_{12}^+\ 2e_{13}^-\ 2e_{13}^+\ 2e_{23}^-\ 2e_{23}^+\ e_{33}]^{\mathrm{T}}$$
$$\boldsymbol{E}=[E_{11}^-\ E_{11}^+\ E_{22}^-\ E_{22}^+\ 2E_{12}^-\ 2E_{12}^+\ 2E_{13}^-\ 2E_{13}^+\ 2E_{23}^-\ 2E_{23}^+\ E_{33}]^{\mathrm{T}}$$
$$\boldsymbol{H}=[H_{11}^-\ H_{11}^+\ H_{22}^-\ H_{22}^+\ H_{12}^-\ H_{12}^+\ H_{13}^-\ H_{13}^+\ H_{23}^-\ H_{23}^+\ H_{33}]^{\mathrm{T}}$$
$$\hat{\boldsymbol{H}}_\Gamma=[\hat{H}_{\nu\nu}^-\ \hat{H}_{\nu\nu}^+\ \hat{H}_{\nu t}^-\ \hat{H}_{\nu t}^+\ \hat{H}_{\nu 3}^-\ \hat{H}_{\nu 3}^+]^{\mathrm{T}}$$
$$\boldsymbol{P}=[-p_1^-\ p_1^+\ -p_2^-\ p_2^+\ -p_3^-\ p_3^+]^{\mathrm{T}}$$

where $\boldsymbol{D}$ is the constitutive stiffness matrix whose components are defined in the next section; $\bar{A}_\alpha = A_\alpha(1+k_\alpha\bar{\delta})$ are the Lamé coefficients of the middle surface $\bar{S}$; $\bar{\delta}=(\delta^-+\delta^+)/2$ is the distance from the reference surface to the middle surface; $v_\nu^\pm$, $v_t^\pm$, and $v_3^\pm$ are the components of the displacement vectors of the face surfaces in the coordinate system ν, t, and α_3 (Figure 1); k_N is the normal curvature of the reference bounding curve Γ; $H_{\alpha\beta}^\pm$, $H_{\alpha3}^\pm$, and H_{33} are the stress resultants; and $\hat{H}_{\nu\nu}^\pm$, $\hat{H}_{\nu t}^\pm$ and $\hat{H}_{\nu3}^\pm$ are the external load resultants given by

$$H_{\alpha\beta}^\pm=\sum_k\int_{\delta_{k-1}}^{\delta_k}\sigma_{\alpha\beta}^{(k)}N^\pm\mathrm{d}\alpha_3 \quad (28a)$$

$$H_{\alpha3}^\pm=\sum_k\int_{\delta_{k-1}}^{\delta_k}\sigma_{\alpha3}^{(k)}N^\pm\mathrm{d}\alpha_3 \quad H_{33}=\sum_k\int_{\delta_{k-1}}^{\delta_k}\sigma_{33}^{(k)}\mathrm{d}\alpha_3 \quad (28b)$$

$$\hat{H}_{\nu æ}^\pm=\sum_k\int_{\delta_{k-1}}^{\delta_k}q_{æ}^{(k)}N^\pm\mathrm{d}\alpha_3 \quad æ=\nu,t,\text{ and }3 \quad (28c)$$

Mixed variational equation (26) will be used in the following work [26] for constructing assumed stress–strain four-node *curved* shell elements.

§4. CONSTITUTIVE EQUATIONS

In this section, four types of the constitutive equations are discussed. We consider first an orthotropic ply and then study the more general case of monoclinic symmetry.

4.1. Complete Constitutive Equations

Consider the kth orthotropic layer of the shell and denote its axes of symmetry as $2_{1'}^{(k)}$, $\alpha_{2'}^{(k)}$, and α_3. In these axes of symmetry, the equations of the complete 3D Hooke law will be

$$\varepsilon_{1'1'}=\frac{1}{E_1^{(k)}}\sigma_{1'1'}^{(k)}-\frac{\nu_{21}^{(k)}}{E_2^{(k)}}\sigma_{2'2'}^{(k)}-\frac{\nu_{31}^{(k)}}{E_3^{(k)}}\sigma_{33}^{(k)}$$
$$\varepsilon_{2'2'}=-\frac{\nu_{12}^{(k)}}{E_1^{(k)}}\sigma_{1'1'}^{(k)}+\frac{1}{E_2^{(k)}}\sigma_{2'2'}^{(k)}-\frac{\nu_{32}^{(k)}}{E_3^{(k)}}\sigma_{33}^{(k)} \quad (29)$$
$$2\varepsilon_{1'2'}=\frac{1}{G_{12}^{(k)}}\sigma_{1'2'}^{(k)} \quad 2\varepsilon_{1'3}=\frac{1}{G_{13}^{(k)}}\sigma_{1'3}^{(k)} \quad 2\varepsilon_{2'3}=\frac{1}{G_{23}^{(k)}}\sigma_{2'3}^{(k)}$$
$$\varepsilon_{33}=-\frac{\nu_{13}^{(k)}}{E_1^{(k)}}\sigma_{1'1'}^{(k)}-\frac{\nu_{23}^{(k)}}{E_2^{(k)}}\sigma_{2'2'}^{(k)}+\frac{1}{E_3^{(k)}}\sigma_{33}^{(k)}$$

where $\sigma_{1'1'}^{(k)}$, $\sigma_{2'2'}^{(k)}$, $\sigma_{33}^{(k)}$ and $\sigma_{1'2'}^{(k)}$, $\sigma_{1'3'}^{(k)}$, $\sigma_{2'3}^{(k)}$ are the normal and shear components of the stress tensor in the $\alpha_{1'}^{(k)}$, $\alpha_{2'}^{(k)}$, and α_3 coordinate system; $E_1^{(k)}$, $E_2^{(k)}$, and $E_3^{(k)}$ are the elastic moduli;

$G_{12}^{(k)}$, $G_{13}^{(k)}$, and $G_{23}^{(k)}$ are the shear moduli; and $\nu_{ij}^{(k)}$ are Poisson's ratios. From reasons of symmetry, we have

$$\nu_{ij}^{(k)}E_j^{(k)} = \nu_{ji}^{(k)}E_i^{(k)} \quad \text{for } i \neq j$$

In coordinate directions α_1, α_2, and α_3, when a case of monoclinic symmetry is realized, the equations of the 3D Hooke law can be represented in the more general form

$$\varepsilon_{ij} = \sum_{\ell,m} A_{ij\ell m}^{(k)}\sigma_{\ell m}^{(k)} \tag{30}$$

where $A_{ij\ell m}^{(k)}$ are the components of the material tensor of the kth layer depending on engineering constants $E_i^{(k)}$, $G_{ij}^{(k)}$, and $\nu_{ij}^{(k)}$, whose expressions can be found in many books (see, e.g., [6, 7]) and are not displayed here.

Solving first three of Eqs. (30) for the in-plane stresses, one can find

$$\sigma_{\alpha\beta}^{(k)} = \sum_{\gamma,\delta} Q_{\alpha\beta\gamma\delta}^{(k)}\varepsilon_{\gamma\delta} - \mu_{\alpha\beta 33}^{(k)}\sigma_{33}^{(k)} \tag{31}$$

where

$$\begin{aligned}
\mu_{\alpha\beta 33}^{(k)} &= \sum_{\gamma,\delta} Q_{\alpha\beta\gamma\delta}^{(k)}A_{\gamma\delta 33}^{(k)} \\
Q_{1111}^{(k)} &= \frac{1}{\Delta_k}\left(A_{2222}^{(k)}A_{1212}^{(k)} - A_{2212}^{(k)}A_{1222}^{(k)}\right) \\
Q_{1122}^{(k)} &= \frac{1}{\Delta_k}\left(A_{1112}^{(k)}A_{1222}^{(k)} - A_{1122}^{(k)}A_{1212}^{(k)}\right) \\
Q_{1112}^{(k)} &= \frac{1}{2\Delta_k}\left(A_{1122}^{(k)}A_{2212}^{(k)} - A_{1112}^{(k)}A_{2222}^{(k)}\right) \\
Q_{2222}^{(k)} &= \frac{1}{\Delta_k}\left(A_{1111}^{(k)}A_{1212}^{(k)} - A_{1112}^{(k)}A_{1211}^{(k)}\right) \\
Q_{2212}^{(k)} &= \frac{1}{2\Delta_k}\left(A_{1112}^{(k)}A_{2211}^{(k)} - A_{1111}^{(k)}A_{2212}^{(k)}\right) \\
Q_{1212}^{(k)} &= \frac{1}{4\Delta_k}\left(A_{1111}^{(k)}A_{2222}^{(k)} - A_{1122}^{(k)}A_{2211}^{(k)}\right) \\
\Delta_k &= A_{1211}^{(k)}\left(A_{1122}^{(k)}A_{2212}^{(k)} - A_{1112}^{(k)}A_{2222}^{(k)}\right) \\
&\quad + A_{1222}^{(k)}\left(A_{1112}^{(k)}A_{2211}^{(k)} - A_{1111}^{(k)}A_{2212}^{(k)}\right) \\
&\quad + A_{1212}^{(k)}\left(A_{1111}^{(k)}A_{2222}^{(k)} - A_{1122}^{(k)}A_{2211}^{(k)}\right)
\end{aligned} \tag{32}$$

Substituting in-plane stresses (31) in the last of Eqs. (30) and solving for the transverse normal stress, one obtains

$$\sigma_{33}^{(k)} = \sum_{\gamma,\delta} C_{33\gamma\delta}^{(k)}\varepsilon_{\gamma\delta} + C_{3333}^{(k)}\varepsilon_{33} \tag{33}$$

where

$$\begin{aligned}
C_{33\alpha\beta}^{(k)} &= -\frac{1}{\Lambda_k}\mu_{33\alpha\beta}^{(k)} \qquad C_{3333}^{(k)} = \frac{1}{\Lambda_k} \\
\mu_{33\alpha\beta}^{(k)} &= \sum_{\gamma,\delta} Q_{\gamma\delta\alpha\beta}^{(k)}A_{33\gamma\delta}^{(k)} \qquad \Lambda_k = -\sum_{\gamma,\delta} A_{33\gamma\delta}^{(k)}\mu_{\gamma\delta 33}^{(k)} + A_{3333}^{(k)}
\end{aligned} \tag{34}$$

Using transverse normal stress (33) in Eq. (31) yields

$$\sigma_{\alpha\beta}^{(k)} = \sum_{\gamma,\delta} C_{\alpha\beta\gamma\delta}^{(k)}\varepsilon_{\gamma\delta} + C_{\alpha\beta 33}^{(k)}\varepsilon_{33} \tag{35}$$

where

$$C_{\alpha\beta\gamma\delta}^{(k)} = Q_{\alpha\beta\gamma\delta}^{(k)} + \frac{1}{\Lambda_k}\mu_{\alpha\beta 33}^{(k)}\mu_{33\gamma\delta}^{(k)} \qquad C_{\alpha\beta 33}^{(k)} = -\frac{1}{\Lambda_k}\mu_{\alpha\beta 33}^{(k)} \tag{36}$$

Finally, one can derive from Hooke law (30) the remaining equations for the transverse shear stresses as

$$\sigma_{\alpha 3}^{(k)} = 2\sum_{\gamma} C_{\alpha 3\gamma 3}^{(k)}\varepsilon_{\gamma 3} \tag{37}$$

where

$$\begin{aligned}
C_{1313}^{(k)} &= \frac{1}{d_k}A_{2323}^{(k)} \quad C_{1323}^{(k)} = -\frac{1}{d_k}A_{1323}^{(k)} \quad C_{2323}^{(k)} = \frac{1}{d_k}A_{1313}^{(k)} \\
d_k &= 4\left(A_{1313}^{(k)}A_{2323}^{(k)} - A_{1323}^{(k)}A_{2313}^{(k)}\right)
\end{aligned} \tag{38}$$

Unfortunately, such a shell formulation on the basis of the complete 3D constitutive law, Eqs. (33)–(38), is deficient because so-called thickness locking [15, 16] can occur. This phenomenon occurs in bending dominated shell problems when Poisson ratios are not equal to zero. The main reason is that Poisson's effect in the thickness direction is taken into account in equations of Hooke law (30) for the in-plane strains.

To avoid thickness locking, the effective remedies may be used.

4.2. Modified Constitutive Equations

It is well known that in the ESL shell theory the constitutive equations for transverse stresses are not satisfied pointwise [6, 10] but may be fulfilled in an integral sense. In particular, for the transverse normal component, the following integral equation [24] should be adopted:

$$\sum_k \int_{\delta_{k-1}}^{\delta_k}\left(\sigma_{33}^{(k)} - \sum_{\gamma,\delta} C_{33\gamma\delta}^{(k)}\varepsilon_{\gamma\delta} - C_{3333}^{(k)}\varepsilon_{33}\right)d\alpha_3 = 0 \tag{39}$$

Taking into account relations (2) and introducing notations

$$H_{33}^{(k)} = \int_{\delta_{k-1}}^{\delta_k}\sigma_{33}^{(k)}d\alpha_3 \qquad n_k^p = \int_{\delta_{k-1}}^{\delta_k}(N^-)^{1-p}(N^+)^p d\alpha_3$$
$$p = 0 \text{ and } 1 \tag{40}$$

one can write Eq. (39) as follows:

$$\sum_k\left[H_{33}^{(k)} - \sum_{\gamma,\delta} C_{33\gamma\delta}^{(k)}\left(n_k^0 e_{\gamma\delta}^- + n_k^1 e_{\gamma\delta}^+\right) - h_k C_{3333}^{(k)}e_{33}\right] = 0 \tag{41}$$

This allows us to assume that the transverse normal stress is independent of the thickness coordinate α_3, which may be appreciated as a good remedy [18–20] for overcoming the thickness locking phenomenon. So, following this idea, the transverse normal stress is assumed to be constant in the thickness direction for every layer of the shell:

$$\sigma_{33}^{(k)} = \frac{1}{h_k} H_{33}^{(k)} = \frac{1}{h_k} \sum_{\gamma,\delta} C_{33\gamma\delta}^{(k)} \left(n_k^0 e_{\gamma\delta}^- + n_k^1 e_{\gamma\delta}^+\right) + C_{3333}^{(k)} e_{33} \quad (42)$$

Substituting further thickness stress (42) in Eq. (31) and integrating these modified equations together with remaining constitutive equations (37) across the shell thickness, and accounting for relations (2), (27), (28), and (34), one derives

$$\boldsymbol{H} = \boldsymbol{D}\boldsymbol{e} \quad (43)$$

where $\boldsymbol{D}$ denotes a constitutive stiffness matrix introduced in Section 3 whose components are defined as

$$D_{\alpha\beta\gamma\delta}^{pq} = \sum_k \left(n_k^{pq} Q_{\alpha\beta\gamma\delta}^{(k)} + \frac{n_k^p n_k^q \mu_{\alpha\beta33}^{(k)} \mu_{33\gamma\delta}^{(k)}}{h_k \Lambda_k} \right) \quad (44)$$

$$D_{\alpha\beta33}^{p} = -\sum_k \frac{1}{\Lambda_k} n_k^p \mu_{\alpha\beta33}^{(k)} \qquad D_{33\alpha\beta}^{p} = -\sum_k \frac{1}{\Lambda_k} n_k^p \mu_{33\alpha\beta}^{(k)}$$

$$D_{\alpha3\beta3}^{pq} = \sum_k n_k^{pq} C_{\alpha3\beta3}^{(k)} \qquad D_{3333} = \sum_k \frac{1}{\Lambda_k} h_k$$

$$n_k^{pq} = \int_{\delta_{k-1}}^{\delta_k} (N^-)^{2-p-q} (N^+)^{p+q} d\alpha_3$$

Throughout this section superscripts p and q take the values 0 and 1.

It should be mentioned that this *modified* constitutive law is quite efficient for most engineering problems. But for incompressible or nearly incompressible materials [16], some difficulties can occur, leading to volumetric locking.

4.3. Reduced Constitutive Equations

To simultaneously circumvent thickness and volumetric locking, we invoke a standard engineering assumption, $\sigma_{33}^{(k)} \ll \sigma_{\alpha\beta}^{(k)}$. This implies that coefficients $A_{\alpha\beta33}^{(k)}$ should be formally set to zero in equations of Hooke's law (30) for the in-plane strains. At the same time, the last equation for the thickness strain is left unchanged (i.e., $A_{33\alpha\beta}^{(k)} \neq 0$). Allowing for this assumption and using a technique described in Section 4.1, one arrives at constitutive equations for in-plane stresses (31), where $\mu_{\alpha\beta33}^{(k)} = 0$ in accordance with Eq. (32), and constitutive equations for transverse stresses (33) and (37).

Integrating these reduced constitutive equations across the shell thickness, and accounting for relations (2), (27), (28), (34), and (43), the following expressions for components of the constitutive stiffness matrix $\boldsymbol{D}$ are obtained:

$$D_{\alpha\beta\gamma\delta}^{pq} = \sum_k n_k^{pq} Q_{\alpha\beta\gamma\delta}^{(k)} \qquad D_{\alpha\beta33}^{p} = 0$$

$$D_{33\alpha\beta}^{p} = -\sum_k \frac{1}{A_{3333}^{(k)}} n_k^p \mu_{33\alpha\beta}^{(k)} \quad (45)$$

$$D_{\alpha3\beta3}^{pq} = \sum_k n_k^{pq} C_{\alpha3\beta3}^{(k)} \qquad D_{3333} = \sum_k \frac{1}{A_{3333}^{(k)}} h_k$$

One can observe that relations (45) are obtained from more general ones, Eqs. (44) when $\mu_{\alpha\beta33}^{(k)} = 0$ and $\Lambda_k = A_{3333}^{(k)}$, according to Eq. (34).

The *reduced* constitutive law was proposed in the literature [24, 25, 28] for overcoming thickness locking and showed a good performance in the case of using the Timoshenko–Mindlin-type shell theory. It should be mentioned that this approach yields the nonsymmetric material matrix and, as a result, more computational efforts have to be made.

4.4. Simplified Constitutive Equations

When a shell has undergone pure bending, one half of the shell body in the thickness direction is under tension and the other half is under compression; that is, the thickness strain according to the complete 3D Hooke law would be zero due to the limitation of the linear displacement approximation (1). Therefore, a shell will be in the plane-strain state instead of the expected plane-stress state. To circumvent these difficulties, the simplified constitutive stiffness matrix may be employed:

$$D_{\alpha\beta\gamma\delta}^{pq} = \sum_k n_k^{pq} Q_{\alpha\beta\gamma\delta}^{(k)} \qquad D_{\alpha\beta33}^{p} = D_{33\alpha\beta}^{p} = 0 \quad (46)$$

$$D_{\alpha3\beta3}^{pq} = \sum_k n_k^{pq} C_{\alpha3\beta3}^{(k)} \qquad D_{3333} = \sum_k \frac{1}{A_{3333}^{(k)}} h_k$$

This is due to the plane-stress enforcement [5, 21–23], which is done by decoupling the transverse normal stress with all other stresses in 3D Hooke law (30); that is, it is supposed that

$$A_{\alpha\beta33}^{(k)} = A_{33\alpha\beta}^{(k)} = 0$$

It is apparent that the *simplified* constitutive law leads to the symmetric constitutive stiffness matrix $\boldsymbol{D}$, but it is slightly deficient for the thick anisotropic shells. So, allowing for the simplicity of such an approach, it may be recommended for an analysis of composite thin-walled structures.

In the companion paper [26], all three remedies for overcoming thickness locking, Eqs. (44)–(46), are evaluated with several discriminating problems selected from the literature.

§5. CONCLUSION

The principally new strain–displacement relationships of the ESL and LW shell theories have been developed. In both theories, the transverse normal deformation response is included. These relationships are very attractive because they are objective (i.e., invariant under all rigid-body motions). Therefore, they may be used for the formulation of effective curved multilayered shell elements in curvilinear reference surface coordinates. However, the practical use of such strain–displacement relationships requires a development of the constitutive equations to overcome thickness and volumetric locking phenomena. For this purpose, three simple and efficient remedies have been introduced into the formulation by using the unified technique.

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