Refined Global Approximation Theory of Multilayered Plates and Shells

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Abstract: A refined global approximation theory of thin multilayered anisotropic shells is developed. The effects of the laminated material response, transverse shear, and transverse normal strains are included. The material of each layer of the shell is assumed to be linearly elastic, anisotropic, homogeneous, or fiber reinforced. As unknown functions the tangential displacements of the face surfaces and transverse displacements of the face or middle surfaces of the shell are chosen. It is an important feature of the proposed theory. This fact simplifies, for example, an analysis of the contact problems and allows to elaborate universal numerical algorithms. An exact solution for the problem of the bending of homogeneous isotropic rectangular plates subjected to a sinusoidal load has been found. Numerical solutions for the problem of the bending of multilayered composite plates and cylindrical shells have been also obtained. The influence of anisotropy, and transverse shear and transverse normal deformation response of the stress state of the shell is examined.

INTRODUCTION

In global approximation theories, global through-the-thickness displacement, strain, or stress approximations are introduced and in a result the multilayered anisotropic shell is replaced by an equivalent single-layer anisotropic shell (Lo et al. 1977; Cohen 1978; Reissner 1979; Reddy 1984, 1997; Noor and Peters 1987; Grigolyuk and Kulikov 1988; Noor and Burton 1990; Kulikov 1996, 2001). Consequently, the order of the governing equations is independent on the number of layers of the shell. The simplest examples of these theories are the first-order shear deformation theories based on the kinematic Timoshenko hypothesis (the linear distribution of displacements in the thickness direction). More general examples of these theories are the so-called higher-order theories based on the nonlinear distribution of displacements in the thickness direction as was suggested by Reissner (1952), Naghdi (1957), and Reddy (1997).

The direct use of the traditional global approximation theories for solving a series of important shell problems such as the contact problems is not always convenient. In these problems it is more convenient to select as unknown functions the tangential and transverse displacements of the face surfaces of the shell, since with the help of these displacements the kinematic requirements of no penetration of the contact bodies can be fulfilled (Kulikov and Plotnikova 1998). Furthermore, the proposed approach can simplify a formulation of nonlinear strain-displacement equations (Kulikov 2001), and an elaboration of new finite-element models (Kulikov and Plotnikova 2001).

Herein, the refined first-order and higher-order global approximation theories of multilayered anisotropic shells are developed. The material of each layer of the shell is assumed to be linearly elastic, anisotropic, homogeneous, or fiber reinforced, such that in each point there is a single surface of elastic symmetry parallel to the reference surface. The governing equations of the theories are obtained by using the principle of the virtual work. An outcome of this approach is that the transverse shear and transverse normal stresses are computed by integrating the equations of the three-dimensional elasticity theory with respect to the normal coordinate. In a result, the transverse stress components satisfy the boundary conditions on the bottom and top surfaces of the shell, and the equilibrium conditions at the layer interfaces.

The proposed theories are used for an analysis of the effects of the transverse normal deformation response and anisotropy in thick plates and cylindrical shells. It is shown that the above effects are significant for the thick cross-ply composite shell structures.

VARIATIONAL EQUATION OF ELASTICITY THEORY

Consider the shell built up in the general case by the arbitrary superposition across the wall thickness of N thin layers of uniform thickness h_k. The kth layer may be defined as a three-dimensional body of the volume V_k bounded by two surfaces S_{k-1} and S_k, located at the distances δ_{k-1} and δ_k measured with respect to the reference surface S, and the edge boundary surface Ω_k that is perpendicular to the reference surface (Fig. 1). It is also assumed that the bounding surfaces and the reference surface are continuous and sufficiently smooth, and without any singularities. The constituent layers of the shell are supposed to be rigidly joined, so that no slip on contact surfaces and no separation of layers can occur. The material of each layer is assumed to be linearly elastic, anisotropic, homogeneous, or fiber reinforced, such that in each point of

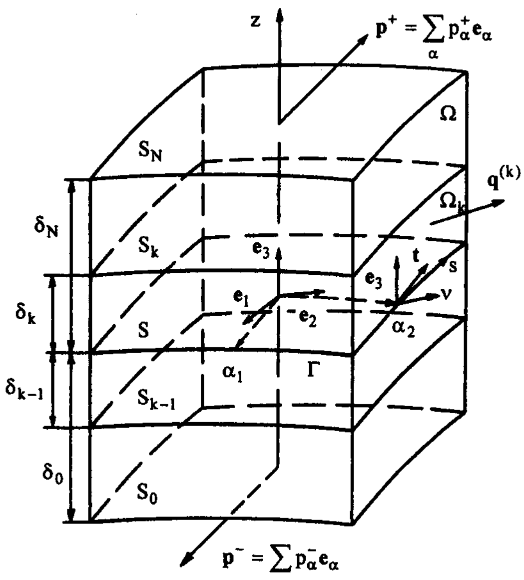

FIG. 1. Element of Multilayered Shell

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the shell there is a single surface of elastic symmetry parallel to the reference surface.

The governing equations of the theory of elastic multilayered anisotropic shells can be obtained by applying the principle of the virtual work. It permits the reduction of the three-dimensional stress-strain state to an equivalent two-dimensional one and can be written in the following form (Washizu 1982):

$$\sum_{k=1}^{N}\int_{V_k}\int\int\sum_{\alpha\leq\beta}\sigma_{\alpha\beta}^{(k)}\delta\varepsilon_{\alpha\beta}^{(k)}A_1A_2\,d\alpha_1\,d\alpha_2\,dz-\int_{S_N}\int\sum_{\alpha}p_{\alpha}^{+}\delta u_{\alpha}^{(N)}\,dS$$
$$+\int_{S_0}\int\sum_{\alpha}p_{\alpha}^{-}\delta u_{\alpha}^{(1)}\,dS+\sum_{n=1}^{N-1}\int_{S_n}\int\sum_{\alpha}\tau_{\alpha}^{(n)}[\delta u_{\alpha}^{(n+1)}-\delta u_{\alpha}^{(n)}]\,dS$$
$$-\sum_{k=1}^{N}\int_{\Omega_k}\int[q_{\nu}^{(k)}\delta u_{\nu}^{(k)}+q_{t}^{(k)}\delta u_{t}^{(k)}+q_{3}^{(k)}\delta u_{3}^{(k)}]\,dS=0 \quad (1)$$

where α_1 and α_2 = orthogonal curvilinear coordinates that coincide with the lines of principal curvatures of the reference surface; z = coordinate normal to the reference surface; k_i and A_i = principal curvatures and Lamé coefficients of the reference surface; p_{α}^{-} and p_{α}^{+} = intensities of the external loading acting on the bottom surface S_0 and top surface S_N in the α_1, α_2, z-coordinate directions; $q_{\nu}^{(k)}$, $q_{t}^{(k)}$, and $q_{3}^{(k)}$ are the intensities of the external loading acting on the edge boundary surface Ω_k in the ν, t, z-directions; $\mathbf{v}$ and $\mathbf{t}$ are the normal and tangential unit vectors to the bounding curve Γ (Fig. 1); $\sigma_{\alpha\beta}^{(k)}$ and $\varepsilon_{\alpha\beta}^{(k)}$ = stresses and strains of the kth layer; $\tau_{\alpha}^{(n)}$ = interlaminar stresses acting on the contact surfaces of the layers; $u_{\alpha}^{(k)}$ = displacement vector components of the kth layer in the α_1, α_2, z-directions; $u_{\nu}^{(k)}$, $u_{t}^{(k)}$, and $u_{3}^{(k)}$ = displacement vector components of the kth layer in the ν, t, z-directions. Here and in the following developments the index k takes the values 1, 2, . . . , N; the index n takes the values 1, 2, . . . , $N-1$; indices i, j take the values 1, 2 and indices α, β take the values 1, 2, 3.

The three-dimensional strain-displacement relations for the thin multilayered shell will be

$$\varepsilon_{11}^{(k)}=\frac{1}{A_1}\frac{\partial u_1^{(k)}}{\partial\alpha_1}+B_2u_2^{(k)}+k_1u_3^{(k)}\ (1\Leftrightarrow 2) \quad (2a)$$

$$\varepsilon_{12}^{(k)}=\frac{1}{A_1}\frac{\partial u_2^{(k)}}{\partial\alpha_1}+\frac{1}{A_2}\frac{\partial u_1^{(k)}}{\partial\alpha_2}-B_2u_1^{(k)}-B_1u_2^{(k)} \quad (2b)$$

$$\varepsilon_{13}^{(k)}=\frac{\partial u_1^{(k)}}{\partial z}+\frac{1}{A_1}\frac{\partial u_3^{(k)}}{\partial\alpha_1}-k_1u_1^{(k)}\ (1\Leftrightarrow 2) \quad (2c)$$

$$\varepsilon_{33}^{(k)}=\frac{\partial u_3^{(k)}}{\partial z} \quad (2d)$$

where $B_1=(1/A_1A_2)(\partial A_2/\partial\alpha_1)$ $(1\Leftrightarrow 2)$ and the sign $(1\Leftrightarrow 2)$ accompanying certain relations means that the remaining relations, not explicitly written, are correspondingly obtained by replacing subscript 1 by 2 and vice-versa. It is important to note that, due to consideration of the thin shell structures, the Lamé coefficients $H_i=A_i(1+k_iz)$ are replaced by their simplified expressions, i.e. the Lamé coefficients of the reference surface A_i.

The governing equations of the elasticity theory for thin multilayered shells can be derived by applying the principal of the virtual work (1). Substituting the strain-displacement relations (2) into (1) and using Gauss's theorem, one can obtain the following variational equation:

$$\sum_{k=1}^{N}\int_{V_k}\int\int\sum_{\alpha}L_{\alpha}^{(k)}\delta u_{\alpha}^{(k)}A_1A_2\,d\alpha_1\,d\alpha_2\,dz$$
$$-\int_{S_N}\int\sum_{\alpha}[\sigma_{\alpha3}^{(N)}-p_{\alpha}^{+}]\delta u_{\alpha}^{(N)}\,dS+\int_{S_0}\int\sum_{\alpha}[\sigma_{\alpha3}^{(1)}-p_{\alpha}^{-}]\delta u_{\alpha}^{(1)}\,dS$$
$$+\sum_{n=1}^{N-1}\int_{S_n}\int\sum_{\alpha}\{[\sigma_{\alpha3}^{(n+1)}-\tau_{\alpha}^{(n)}]\delta u_{\alpha}^{(n+1)}-[\sigma_{\alpha3}^{(n)}-\tau_{\alpha}^{(n)}]\delta u_{\alpha}^{(n)}\}\,dS$$
$$-\sum_{k=1}^{N}\int_{\Omega_k}\int\{[\sigma_{\nu\nu}^{(k)}-q_{\nu}^{(k)}]\delta u_{\nu}^{(k)}+[\sigma_{\nu t}^{(k)}-q_{t}^{(k)}]\delta u_{t}^{(k)}$$
$$+[\sigma_{\nu3}^{(k)}-q_{3}^{(k)}]\delta u_{3}^{(k)}\}\,dS=0 \quad (3)$$

where $\sigma_{\nu\nu}^{(k)}$, $\sigma_{\nu t}^{(k)}$, and $\sigma_{\nu3}^{(k)}$ are the stress components of the kth layer in the coordinate system ν, t, z; $L_{\alpha}^{(k)}$ are the differential operators of the three-dimensional elasticity theory corresponding to the adopted strain-displacement relations (2):

$$L_1^{(k)}=\frac{1}{A_1}\frac{\partial\sigma_{11}^{(k)}}{\partial\alpha_1}+\frac{1}{A_2}\frac{\partial\sigma_{12}^{(k)}}{\partial\alpha_2}+\frac{\partial\sigma_{13}^{(k)}}{\partial z}+B_1[\sigma_{11}^{(k)}-\sigma_{22}^{(k)}]+2B_2\sigma_{12}^{(k)}$$
$$+k_1\sigma_{13}^{(k)}\ (1\Leftrightarrow 2) \quad (4a)$$

$$L_3^{(k)}=\frac{1}{A_1}\frac{\partial\sigma_{13}^{(k)}}{\partial\alpha_1}+\frac{1}{A_2}\frac{\partial\sigma_{23}^{(k)}}{\partial\alpha_2}+\frac{\partial\sigma_{33}^{(k)}}{\partial z}+B_1\sigma_{13}^{(k)}+B_2\sigma_{23}^{(k)}$$
$$-k_1\sigma_{11}^{(k)}-k_2\sigma_{22}^{(k)} \quad (4b)$$

FUNDAMENTAL EQUATIONS OF ELASTICITY THEORY FOR MULTILAYERED SHELLS

Equating to zero the coefficients at arbitrary variations of displacements, the fundamental equations of the elasticity theory for the thin multilayered shell of the uniform thickness are obtained:

- The equilibrium equations for the kth layer

$$L_{\alpha}^{(k)}=0 \quad (5)$$

- The boundary conditions for the transverse stress components on the top surface S_N

$$\sigma_{\alpha3}^{(N)}=p_{\alpha}^{+} \quad (6)$$

- The boundary conditions for the transverse stress components on the bottom surface S_0

$$\sigma_{\alpha3}^{(1)}=p_{\alpha}^{-} \quad (7)$$

- The equilibrium conditions for the transverse stress components at the layer interfaces S_n

$$\sigma_{\alpha3}^{(n+1)}=\sigma_{\alpha3}^{(n)} \quad (8)$$

- The boundary conditions on the edge boundary surfaces Ω_k

$$\sigma_{\nu\nu}^{(k)}=q_{\nu}^{(k)},\quad \sigma_{\nu t}^{(k)}=q_{t}^{(k)}\quad\text{and}\quad \sigma_{\nu3}^{(k)}=q_{3}^{(k)} \quad (9)$$

Additionally we should invoke the stress-strain relations

$$\sigma_{11}^{(k)}=C_{11}^{(k)}\varepsilon_{11}^{(k)}+C_{12}^{(k)}\varepsilon_{22}^{(k)}+\underline{C_{13}^{(k)}\varepsilon_{33}^{(k)}}+C_{16}^{(k)}\varepsilon_{12}^{(k)} \quad (10a)$$

$$\sigma_{22}^{(k)}=C_{12}^{(k)}\varepsilon_{11}^{(k)}+C_{22}^{(k)}\varepsilon_{22}^{(k)}+\underline{C_{23}^{(k)}\varepsilon_{33}^{(k)}}+C_{26}^{(k)}\varepsilon_{12}^{(k)} \quad (10b)$$

$$\sigma_{33}^{(k)}=C_{13}^{(k)}\varepsilon_{11}^{(k)}+C_{23}^{(k)}\varepsilon_{22}^{(k)}+C_{33}^{(k)}\varepsilon_{33}^{(k)}+C_{36}^{(k)}\varepsilon_{12}^{(k)} \quad (10c)$$

$$\sigma_{12}^{(k)}=C_{16}^{(k)}\varepsilon_{11}^{(k)}+C_{26}^{(k)}\varepsilon_{22}^{(k)}+\underline{C_{36}^{(k)}\varepsilon_{33}^{(k)}}+C_{66}^{(k)}\varepsilon_{12}^{(k)} \quad (10d)$$

$$\sigma_{23}^{(k)}=C_{44}^{(k)}\varepsilon_{23}^{(k)}+C_{45}^{(k)}\varepsilon_{13}^{(k)} \quad (10e)$$

$$\sigma_{13}^{(k)}=C_{45}^{(k)}\varepsilon_{23}^{(k)}+C_{55}^{(k)}\varepsilon_{13}^{(k)} \quad (10f)$$

where $C_{\ell m}^{(k)}$ are the stiffness coefficients of the kth layer (ℓ, m = 1, 2, . . . , 6).

So, we have all fundamental equations, (2), (5)–(10), for finding the stress-strain state of the thin multilayered anisotropic shell.

TWO-DIMENSIONAL THEORIES FOR MULTILAYERED SHELLS

Herein we consider three refined Timoshenko-Mindlin-type (TMT) theories of elastic multilayered anisotropic shells [see Timoshenko (1921) and Mindlin (1951)]. These theories are based on the linear approximation for the tangential displacements in the thickness direction

$$u_i^{(k)} = N^-(z)v_i^- + N^+(z)v_i^+ \tag{11}$$

where $v_i^-(\alpha_1, \alpha_2)$ and $v_i^+(\alpha_1, \alpha_2)$ = tangential displacements of the bottom surface S_0 and top surface S_N; $N^-(z) = (\delta_N - z)/h$ and $N^+(z) = (z - \delta_0)/h$ = linear shape functions; h = total thickness of the shell. The linear approximation (11) may be considered as a refined kinematic Timoshenko hypothesis [see, for example, works by Grigolyuk and Kulikov (1988) or by Noor and Burton (1999), where as unknown functions, the displacements of the reference surface and rotation components are selected]. The advantage of the proposed approach is obvious, since with the help of the displacements v_i^- and v_i^+ the kinematic boundary conditions on the face surfaces of the shell, and in particular, the conditions of no penetration of the contact bodies can be formulated (Kulikov and Plotnikova 1998). Besides, this provides a convenient way to express the nonlinear strain-displacement relations in terms of face surface strains (Kulikov 2001).

However, in each of the developed TMT shell theories the different approximations for the transverse displacement are used.

1. The TMT110 theory is based on the simplest through-the-thickness approximation for the transverse displacement

$$u_3^{(k)} = v_3^0 \tag{12}$$

where $v_3^0(\alpha_1, \alpha_2)$ is the transverse displacement of the middle surface.

2. The TMT111 theory is based on the refined Timoshenko hypothesis adopted for the transverse displacement

$$u_3^{(k)} = N^-(z)v_3^- + N^+(z)v_3^+ \tag{13}$$

where $v_3^-(\alpha_1, \alpha_2)$ and $v_3^+(\alpha_1, \alpha_2)$ are the transverse displacements of the bottom and top surfaces; $N^-(z)$ and $N^+(z)$ are the linear shape functions.

3. The TMT112 theory is based on the refined Reissner (1952) hypothesis adopted for the transverse displacement

$$u_3^{(k)} = L^-(z)v_3^- + L^0(z)v_3^0 + L^+(z)v_3^+ \tag{14}$$

where $v_3^-(\alpha_1, \alpha_2)$, $v_3^0(\alpha_1, \alpha_2)$, and $v_3^+(\alpha_1, \alpha_2)$ are the transverse displacements of the bottom, middle, and top surfaces, respectively; $L^-(z) = N^-(z)[N^-(z) - N^+(z)]$, $L^0(z) = 4N^-(z)N^+(z)$, and $L^+(z) = N^+(z)[N^+(z) - N^-(z)]$ are the quadratic shape functions.

TMT112 THEORY OF MULTILAYERED ANISOTROPIC SHELLS

With the help of the variational equation, (3), we can develop any TMT theory of multilayered anisotropic shells. Here, for conciseness, we will consider only one theory, namely, TMT112 shell theory which will demonstrate the feature of our approach.

The equations of this higher-order theory of thin multilayered anisotropic shells will be derived by adopting the following basic assumptions:

- The tangential displacements are distributed over the shell thickness according to the refined Timoshenko hypothesis (11).
- The transverse displacement is distributed over the shell thickness according to the refined Reissner hypothesis (14).
- The displacements of the shell are assumed to be small.

Substituting the displacements from (11) and (14) into the strain-displacement relations (2), one can derive the strain-displacement relations of the TMT112 shell theory

$$\varepsilon_{i\alpha}^{(k)} = L^-(z)E_{i\alpha}^- + L^0(z)E_{i\alpha}^0 + L^+(z)E_{i\alpha}^+ \tag{15a}$$

$$\varepsilon_{33}^{(k)} = N^-(z)E_{33}^- + N^+(z)E_{33}^+ \tag{15b}$$

where $E_{\alpha\beta}^-$, $E_{\alpha\beta}^0$, and $E_{\alpha\beta}^+$ are the strains of the bottom, middle, and top surfaces of the shell, respectively

$$E_{11}^s = \frac{1}{A_1}\frac{\partial v_1^s}{\partial \alpha_1} + B_2 v_2^s + k_1 v_3^s \ (1 \Leftrightarrow 2) \tag{16a}$$

$$E_{12}^s = \frac{1}{A_1}\frac{\partial v_2^s}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial v_1^s}{\partial \alpha_2} - B_2 v_1^s - B_1 v_2^s \tag{16b}$$

$$E_{i3}^s = \beta_i - \theta_i^s \tag{16c}$$

$$E_{33}^- = \frac{4}{h}(v_3^0 - v_3^-) - \beta_3 \quad \text{and} \quad E_{33}^+ = \frac{4}{h}(v_3^+ - v_3^0) - \beta_3 \tag{16d}$$

$$\theta_i^s = k_i v_i^s - \frac{1}{A_i}\frac{\partial v_3^s}{\partial \alpha_i} \tag{16e}$$

$$\beta_\alpha = \frac{1}{h}(v_\alpha^+ - v_\alpha^-) \tag{16f}$$

$$v_i^0 = \frac{1}{2}(v_i^- + v_i^+) \tag{16g}$$

Here and in the following developments the index s takes the values −, +, and 0.

Substituting again the displacements from (11) and (14) into the variational equation, (3), and equating to zero the coefficients at arbitrary variations of displacements, one can obtain the following original relationships of the theory of thin multilayered shells:

- The equilibrium equations

$$\sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} L_i^{(k)} N^{\pm}(z)\, dz = 0 \tag{17a}$$

$$\sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} L_3^{(k)} L^s(z)\, dz = 0 \tag{17b}$$

- The boundary conditions for the transverse stress components on the top surface (6).
- The boundary conditions for the transverse stress components on the bottom surface (7).
- The equilibrium conditions for the transverse stress components at the layer interfaces (8).
- The natural boundary conditions on the edge boundary surface $\Omega = \Omega_1 + \Omega_2 + \cdots + \Omega_N$.

$$(H_{\nu\nu}^{\pm} - \hat{H}_{\nu\nu}^{\pm})\delta v_\nu^{\pm} = 0 \quad \text{and} \quad (H_{\nu t}^{\pm} - \hat{H}_{\nu t}^{\pm})\delta v_t^{\pm} = 0 \tag{18a}$$

$$(Q^s_{v3} - \hat{Q}^s_{v3})\delta v^s_3 = 0 \quad (18b)$$

where $H^{\pm}_{\nu\nu}$, $H^{\pm}_{\nu t}$, and $Q^s_{\nu 3}$ are the generalized stress resultants

$$H^{\pm}_{\nu\nu} = \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} \sigma^{(k)}_{\nu\nu} N^{\pm}(z)\, dz \quad \text{and} \quad H^{\pm}_{\nu t} = \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} \sigma^{(k)}_{\nu t} N^{\pm}(z)\, dz \quad (19a)$$

$$Q^s_{\nu 3} = \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} \sigma^{(k)}_{\nu 3} L^s(z)\, dz \quad (19b)$$

The generalized load resultants $\hat{H}^{\pm}_{\nu\nu}$, $\hat{H}^{\pm}_{\nu t}$, and $\hat{Q}^s_{\nu 3}$ are obtained from (19) by replacing the stresses $\sigma^{(k)}_{\nu\nu}$, $\sigma^{(k)}_{\nu t}$, and $\sigma^{(k)}_{\nu 3}$ by intensities of the external loading $q^{(k)}_{\nu}$, $q^{(k)}_{t}$, and $q^{(k)}_{3}$, correspondingly.

Integrating (17) across the shell thickness with account of the boundary conditions (6), (7), and equilibrium conditions (8), we obtain seven equilibrium equations of the thin multilayered shell in terms of stress resultants

$$\frac{1}{A_1}\frac{\partial H^{\pm}_{11}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial H^{\pm}_{12}}{\partial \alpha_2} + B_1(H^{\pm}_{11} - H^{\pm}_{22}) + 2B_2 H^{\pm}_{12} + k_1 H^{\pm}_{13}$$

$$\mp \frac{1}{h} T_{13} \pm p^{\pm}_1 = 0 \; (1 \Leftrightarrow 2) \quad (20a)$$

$$\frac{1}{A_1}\frac{\partial Q^{\pm}_{13}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial Q^{\pm}_{23}}{\partial \alpha_2} + B_1 Q^{\pm}_{13} + B_2 Q^{\pm}_{23} - k_1 Q^{\pm}_{11} - k_2 Q^{\pm}_{22}$$

$$\mp \frac{1}{h}(3H^{\pm}_{33} - H^{\mp}_{33}) \pm p^{\pm}_3 = 0 \quad (20b)$$

$$\frac{1}{A_1}\frac{\partial Q^{0}_{13}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial Q^{0}_{23}}{\partial \alpha_2} + B_1 Q^{0}_{13} + B_2 Q^{0}_{23} - k_1 Q^{0}_{11} - k_2 Q^{0}_{22}$$

$$+ \frac{4}{h}(H^{+}_{33} - H^{-}_{33}) = 0 \quad (20c)$$

where T_{i3} = classical stress resultants; $H^{\pm}_{\alpha\beta}$ and $Q^s_{i\alpha}$ = generalized stress resultants

$$T_{i3} = \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} \sigma^{(k)}_{i3}\, dz \quad (21a)$$

$$H^{\pm}_{\alpha\beta} = \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} \sigma^{(k)}_{\alpha\beta} N^{\pm}(z)\, dz \quad (21b)$$

$$Q^s_{i\alpha} = \sum_{k=1}^{N} \int_{\delta_{k-1}}^{\delta_k} \sigma^{(k)}_{i\alpha} L^s(z)\, dz \quad (21c)$$

In order to obtain the constitutive equations for the multilayered anisotropic shell, the stress-strain relations (10) should be used. Unfortunately, such approach cannot correctly describe the shells made of incompressible materials or almost incompressible materials having Poisson's coefficients $\nu_{\alpha\beta} \approx 0{,}5$ ($\alpha \neq \beta$). To avoid this contradiction we should simplify the stress-strain relations (10) for the tangential stress components omitting the underlined terms. It is an acceptable assumption for the thin shell structures as the transverse normal stress $\sigma^{(k)}_{33}$ is negligibly small in comparison with the other stress components. Note, that expressions for the components of the stiffness matrix $C^{(k)}_{\ell m}$ can be found in the paper by Kulikov and Plotnikova (1999).

It is apparent that using (10) for the transverse stress components can lead to inconsistencies, since these components do not satisfy the boundary conditions on the face surfaces of the shell and the equilibrium conditions at the layer interfaces. To avoid confusions, we should integrate the equations of the three-dimensional elasticity theory (5) across the shell thickness with account of the boundary conditions (7) and the equilibrium conditions (8). In a result, the following expressions for the transverse stress components are obtained:

$$\sigma^{(k)}_{13} = p^{-}_1 - \frac{1}{A_1}\frac{\partial P^{(k)}_{11}}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial P^{(k)}_{12}}{\partial \alpha_2} - B_1[P^{(k)}_{11} - P^{(k)}_{22}] - 2B_2 P^{(k)}_{12}$$

$$- k_1 P^{(k)}_{13} \; (1 \Leftrightarrow 2) \quad (22a)$$

$$\sigma^{(k)}_{33} = p^{-}_3 - \frac{1}{A_1}\frac{\partial P^{(k)}_{13}}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial P^{(k)}_{23}}{\partial \alpha_2} - B_1 P^{(k)}_{13} - B_2 P^{(k)}_{23} + k_1 P^{(k)}_{11}$$

$$+ k_2 P^{(k)}_{22} \quad (22b)$$

where

$$P^{(k)}_{i\alpha} = \sum_{n=1}^{k-1} \int_{\delta_{n-1}}^{\delta_n} \sigma^{(n)}_{i\alpha}\, dz + \int_{\delta_{k-1}}^{z} \sigma^{(k)}_{i\alpha}\, dz \quad (23)$$

It should be noted that from (20)–(23), it follows that the boundary conditions on the top surface (6) are also satisfied, since $P^{(N)}_{i\alpha}(\delta_N) = H^{-}_{i\alpha} + H^{+}_{i\alpha}$ and $P^{(N)}_{i\alpha}(\delta_N) = Q^{-}_{i\alpha} + Q^{0}_{i\alpha} + Q^{+}_{i\alpha}$.

So, all governing equations of the refined higher-order theory of elastic multilayered anisotropic shells have been derived. This theory based on the variational approach can be appreciated as one of a chapter of the three-dimensional elasticity theory.

TMT112 THEORY OF ISOTROPIC PLATES

Let us consider a rectangular isotropic plate with elastic constants E and ν with the help of which we will demonstrate the features of our approach, where E is the elastic module and ν is the Poisson coefficient. The plate is subjected to normally distributed loads p^{+}_3 and p^{-}_3. The Cartesian coordinates x and y are located in the middle plane of the plate, the z-axis is oriented along the perpendicular. It is assumed that $0 \leq x \leq a$ and $0 \leq y \leq b$.

Using an approach elaborated by Grigolyuk and Kulikov (1988), one can derive a system of the governing differential equations for finding the displacements of face and middle planes

$$2\frac{\partial^2 v^0_1}{\partial x^2} + (1 - \nu)\frac{\partial^2 v^0_1}{\partial y^2} + (1 + \nu)\frac{\partial^2 v^0_2}{\partial x \partial y} = 0$$

$$(1 \Leftrightarrow 2 \quad \text{and} \quad x \Leftrightarrow y) \quad (24a)$$

$$D\Delta\Delta w = -\frac{h^2}{6(1-\nu)}\Delta(p^{+}_3 - p^{-}_3) + p^{+}_3 - p^{-}_3 \quad (24b)$$

$$\Delta\varphi = \frac{12}{h^2}\varphi \quad (24c)$$

$$\Delta\beta_3 - \lambda\beta_3 - \frac{\nu\lambda}{1-\nu}\left(\frac{\partial v^0_1}{\partial x} + \frac{\partial v^0_2}{\partial y}\right) = -\frac{\lambda}{2E}(p^{+}_3 + p^{-}_3) \quad (24d)$$

$$\Delta v^0_3 - 5\lambda v^0_3 - \frac{1 + 4\nu + 5\nu^2}{1-\nu}\Delta w + 5\lambda w$$

$$= \frac{5(1+\nu)(1+\nu+2\nu^2)}{(1-\nu)Eh}(p^{+}_3 - p^{-}_3) \quad (24e)$$

where $D = Eh^3/[12(1-\nu^2)]$; $\lambda = 24(1+\nu)/h^2$; $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator; and w, φ, and θ = new governing functions:

$$w = \frac{1}{6}(v^{-}_3 + 4v^0_3 + v^{+}_3) \quad (25a)$$

$$\beta_1 = \frac{\partial\theta}{\partial x} + \frac{\partial\varphi}{\partial y} \quad \text{and} \quad \beta_2 = \frac{\partial\theta}{\partial y} - \frac{\partial\varphi}{\partial x} \quad (25b)$$

TABLE 1. Distribution of Central Transverse Displacement U_3 over Thickness of Square Plate for $\nu = 0.3$ and Various Aspect Ratios a/h, Where $U_3 = v_3^0 E/hp_0$; $\Delta = [(U_3^e - U_3^a)/U_3^e] \cdot 100\%$; U_3^e and U_3^a Are Exact and Approximate Solutions

		Elasticity Theory		TMT112 Plate Theory		TMT111 Plate Theory		TMT110 Plate Theory		Reissner Plate Theory		Kirchhoff Plate Theory	
Variant (1)	z/h (2)	U_3 (3)	Δ (4)	U_3 (5)	Δ (6)	U_3 (7)	Δ (8)	U_3 (9)	Δ (10)	U_3 (11)	Δ (12)	U_3 (13)	Δ (14)
$a/h = 2$	−0.5	1.215	—	1.195	1.6	1.191	2.0	0.975	19.8	1.081	11.0	0.448	63.0
$a/h = 2$	0.0	0.967	—	0.973	−0.6	0.975	−0.8	0.975	−0.8	1.081	−11.8	0.448	53.7
$a/h = 2$	0.5	0.772	—	0.764	1.0	0.759	1.7	0.975	−26.3	1.081	−40.0	0.448	42.0
$a/h = 3$	−0.5	3.600	—	3.596	0.1	3.690	−2.5	3.456	4.0	3.692	−2.6	2.270	36.9
$a/h = 3$	0.0	3.491	—	3.502	−0.3	3.456	1.0	3.456	1.0	3.692	−5.8	2.270	35.0
$a/h = 3$	0.5	3.140	—	3.129	0.4	3.222	−2.6	3.456	−10.1	3.692	−17.6	2.270	27.7

$$\theta = -\frac{h^2}{6(1-\nu)}\Delta w - w - \frac{(1+\nu)h}{3(1-\nu)E}(p_3^+ - p_3^-) \quad (25c)$$

Note that (24*a*)–(24*d*) coincide with the similar equations of the TMT111 plate theory replacing $w = (v_3^- + v_3^+)/2$. While (24*e*) is new and allows one to take into account the nonlinear dependence of the transverse displacement on the thickness direction. Besides, the governing equations of the TMT110 plate theory are correspondingly obtained from (24*a*)–(24*c*) by replacing $w = v_3^0$. It should be noted also that (24*c*) describes the well-known Reissner's (1944) edge effect.

As an example consider a supported rectangular plate subjected to the sinusoidal loading $p_3^- = -p_0 \sin(\pi x/a)\sin(\pi y/b)$, $p_3^+ = 0$. We will search a solution of the problem as

$$v_1^\pm = v_{10}^\pm \cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \quad \text{and} \quad v_2^\pm = v_{20}^\pm \sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \quad (26a)$$

$$v_3^i = v_{30}^i \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \quad (26b)$$

In Table 1 the values of the dimensionless transverse displacement U_3 in a center of a square plate ($a = b$) for various values of the transverse coordinate z are presented. The mechanical and geometrical parameters of a plate are taken following: $\nu = 0.3$, $a/h = 2$ and 3. A comparison with exact solutions on the basis of the elasticity theory (Vlasov 1957), Reissner plate theory (Grigolyuk and Kulikov 1988), and classical Kirchhoff plate theory (Timoshenko and Woinowsky-Krieger 1959) is given.

It can be seen that neglecting the transverse normal deformation in the theory of thick plates leads to the significant errors for the transverse displacement. At the same time the TMT112 plate theory only insignificantly updates the results in a comparison with the TMT111 plate theory. Thus, the TMT111 plate theory it is possible to recommend for the use in engineering calculations. We should also mention that for the thin isotropic plates an account of the transverse normal strain is unimportant.

ANALYSIS OF COMPOSITE PLATES

As a numerical example we consider a linear response of a three-layered composite plate (Fig. 2). It is assumed that each layer possesses a single plane of elastic symmetry parallel to the middle plane. The plate is simply supported on the ends $x = 0$ and $x = \ell$, and is subjected to the normal loading $p_3^+ = q \sin \pi x/\ell$. Let us consider the class of problems known as a cylindrical bending, where all components of the displacement vector, and the strain and stress tensors are dependent only on the x- and z-coordinates. The material characteristics of each layer were taken to be those typical of a high modulus graphite-epoxy composite (Pagano 1970): $E_L = 25E$, $E_T = E_Z = E$, $G_{LT} = G_{LZ} = 0.5E$, $G_{TZ} = 0.2E$, $\nu_{LT} = \nu_{LZ} = \nu_{TZ} = 0.25$, where the subscripts of L, T, and Z refer to the longitudinal, transverse, and thickness directions of the individual ply, respectively, and $E = 6896$ MPa. The ply thicknesses and ply orientations, respectively, are ($h/4$, $h/2$, $h/4$) and (γ, $-\gamma$, γ), where γ is measured in the clockwise direction from x to the fiber direction. Fig. 3 shows the distribution of the transverse shear stresses in the thickness direction at the cross section (at $x = 0$) for the ply angle $\gamma = 30°$ and the dimensionless parameter $\xi = 4$ and $\xi = 10$, where $\xi = \ell/h$. The solid curves display the results obtained by using (22) while curves marked by ▲ show the results found with the help of (10). Pagano's (1970) exact solution is denoted by curves marked by ●. It is seen that the proposed equations, (22), give acceptable results for the moderately thick composite plates.

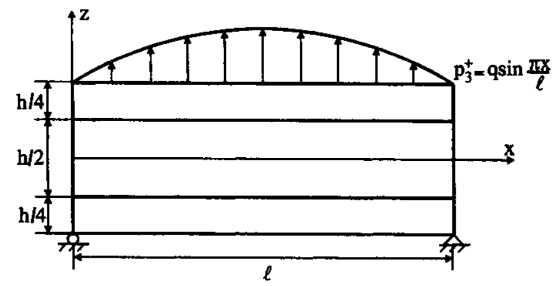

FIG. 2. Cylindrical Bending of Three-Layered Composite Plate

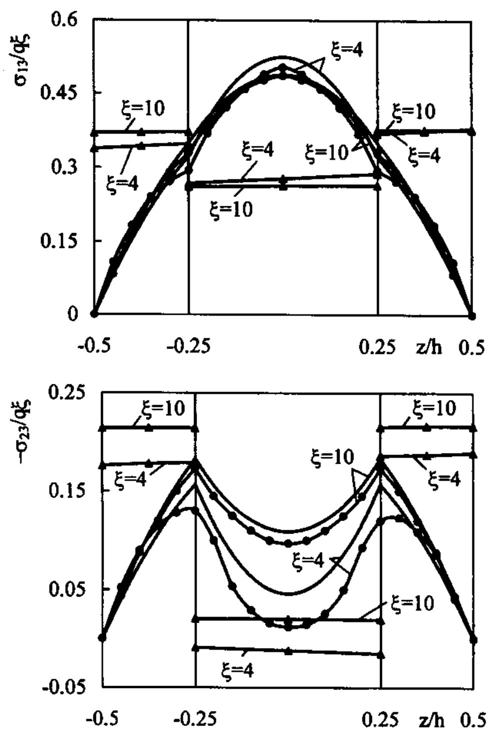

FIG. 3. Distribution of Transverse Shear Stresses σ_{13} and σ_{23} in Thickness Direction at $x = 0$: Eq. (22) (——); Eq. (10) (—▲—); Pagano (—●—)

ANALYSIS OF COMPOSITE CYLINDRICAL SHELLS

To evaluate the influence of the transverse normal strain and anisotropy on the stress-strain state, the axisymmetric multilayered composite cylindrical shell subjected to uniform end stretching is considered (Fig. 4). A numerical solution for this problem based on the Reissner-type shell theory was given by Grigolyuk and Kulikov (1988). The material characteristics of each layer were taken to be those typical of a high modulus graphite-epoxy composite (see the previous section). The geometrical characteristics of the shell are $R = \ell = 100$ mm and $h = 5$ mm. The shell is assumed to be rigidly clamped at $x = 0$ and $x = 100$ mm.

Three groups of problems are treated:

1. The two-layered shell where layer orientations and thicknesses are $(\gamma, -\gamma)$ and $(h/2, h/2)$.
2. The three-layered shell where layer orientations and thicknesses are $(\gamma, -\gamma, \gamma)$ and $(h/4, h/2, h/4)$.
3. The four-layered shell where layer orientations and thicknesses are $(\gamma, -\gamma, \gamma, -\gamma)$ and $(h/4, h/4, h/4, h/4)$.

Fig. 5 shows the dependence of the thickness variation $\Delta h = v_3^+ - v_3^-$ on the ply angle γ in the middle of the shell (at $x = 50$ mm) for various values of a number of layers $N = 2$, 3, and 4. As can be seen, the response of the shell is very unusual to the region $20° < \gamma < 40°$ especially for the two-layered shell, where the thickness variation Δh is positive and has a maximum value. Fig. 6 additionally presents the distribution of the transverse stress components in the thickness direction at the cross section (at $x = 90$ mm) for the ply angle $\gamma = 30°$ and $N = 2$, 3, and 4. Let us pay attention to the same order of the transverse shear stresses σ_{13} and σ_{23} that it points to an essential influence of anisotropy on the stress state in cross-ply composite shells.

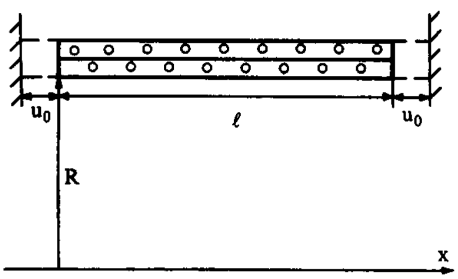

FIG. 4. Multilayered Composite Cylindrical Shell Subjected to Uniform End Stretching

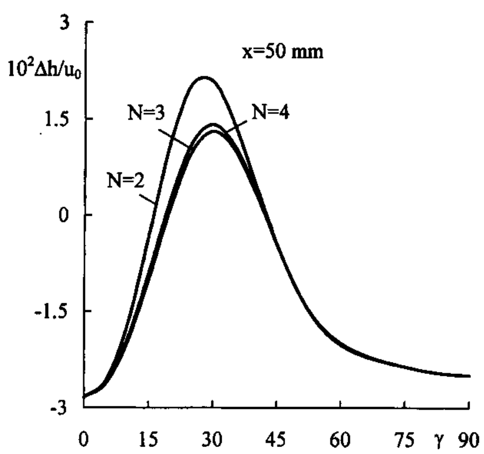

FIG. 5. Dependence of Thickness Variation on Ply Angle

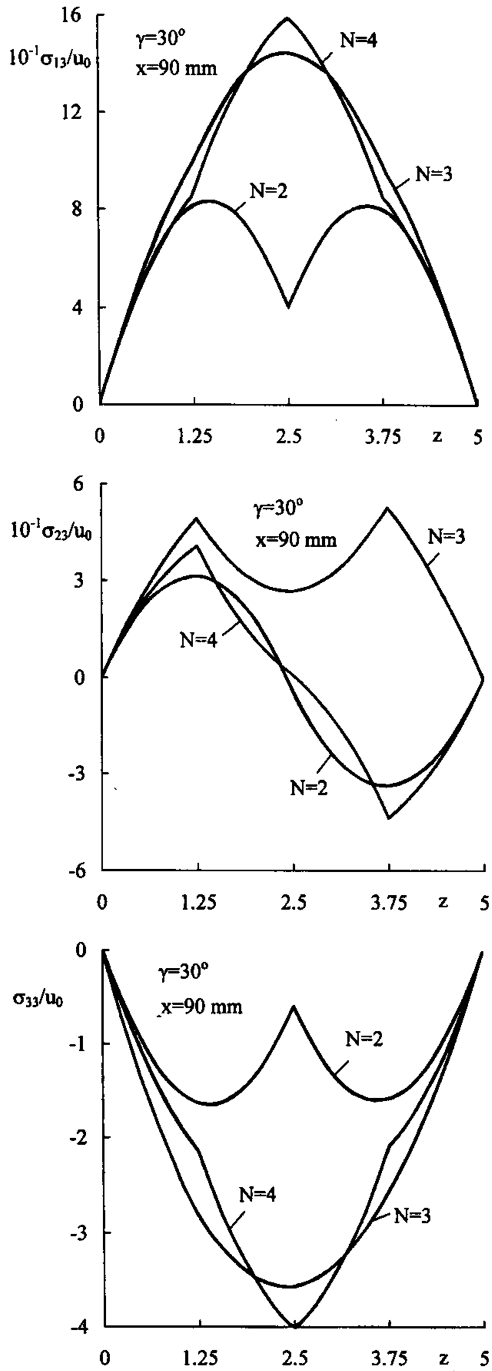

FIG. 6. Distribution of Transverse Stress Components σ_{13}, σ_{23}, and σ_{33} in Thickness Direction

CONCLUSIONS

Three mathematical models for multilayered elastic anisotropic shells have been used here, namely; the TMT110 theory, the TMT111 theory, and the TMT112 theory. One of them, the higher-order TMT112 theory, has been discussed in detail. This theory is based on the refined kinematic Timoshenko and Reissner hypotheses adopted for the tangential and transverse components of the displacement vector, respectively. The effects of laminated material response, transverse shear, and transverse normal strains are included. The material of each layer is assumed to be linearly elastic, anisotropic, homogeneous, or fiber reinforced, such that in each point of the shell there is a single surface of elastic symmetry parallel to the reference surface.

As unknown functions the tangential displacements of the face surfaces and transverse displacements of the face or middle surfaces of the shell have been chosen. Such choice of unknown functions allows one as much as possible to algorithmize the computational modeling of a series of important problems for multilayered shells. The governing equations of the theories have been obtained by applying the principle of the virtual work. An outcome of this approach is that the transverse stress components are obtained by integrating the equations of the three-dimensional elasticity theory with respect to the normal coordinate. As a result, the transverse stress components satisfy the boundary conditions on the face surfaces of the shell and the equilibrium conditions at the layer interfaces.

Using the proposed theories, an influence of the transverse normal deformation on the transverse displacement distribution and thickness variation in isotropic and composite plates, and composite cylindrical shells has been investigated. A comparative analysis on the basis of the above theories has been also given. It has been shown that for the thick shell structures an account of the transverse normal strain is significant. This approach also shows that neglecting the effect of anisotropy can lead to an incorrect description of the stress state in cross-ply composite plates and shells.

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