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Analysis of initially stressed multilayered shells

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Abstract

A refined non-linear first-order global approximation theory of initially stressed multilayered composite shells is developed. The material of each layer of the shell is assumed to be linearly elastic, anisotropic, homogeneous or fiber reinforced. The transverse shear and transverse normal effects are included. It is also assumed that the well-known three-dimensional partially non-linear Novozhilov's strain–displacement relationships are valid. As unknown functions, the tangential and transverse displacements of the top and bottom surfaces of the shell are selected. The paper focuses on two computational models for solving the non-linear problems of prestressed multilayered shells, namely, the axisymmetric deformation of initially stressed multilayered composite shells of revolution and non-axisymmetric deformation of these shells. The joint influence of anisotropy, initially stressed state response, geometrical non-linearity, transverse shear and transverse normal deformation response on the stress state of the shell is examined. It is shown that neglecting the effects of anisotropy and geometrical non-linearity leads to an incorrect description of the stress field in multilayered toroidal shells made of cord–rubber composites. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Elasticity; Anisotropy; Composite shell; Large displacements

1. Introduction

In recent years, considerable interest has been found in the concerned literature to substantiate the geometrically non-linear theory of elastic multilayered composite shells and plates. In this context, there are a number of monographs and survey papers indicated (Bogdanovich, 1987; Grigolyuk and Kulikov, 1988a,b; Libresku, 1975; Noor and Burton, 1990; Reddy, 1997), where rich references of the literature dealing with similar problems to the ones in our study can be found. For some works addressing the problem of multilayered composite plates and shells under initial stresses, and especially with application to tires, the reader is referred to (Biot, 1974; Grigolyuk and Kulikov, 1988a,b, 1992; Kulikov, 1990, 1996, 2001; Noor et al. 1993; Sun and Whitney, 1976; Tanner et al. 1994).

Pneumatic tires are the most widely used composite structures of commercial importance today. Pneumatic tires demand the careful investigation of their strength at the designing stage, which requires the

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development of mathematical models, computational algorithms and computer programs for tires under different types of loading. It is necessary to emphasize that satisfactorily solving the strength problems of pneumatic tires is only possible on the basis of a theory taking into account the spatial character of the stress–strain fields, the effects of anisotropy and the geometrical non-linearity in tires modeled by multilayered shells of revolution with complicated shapes. The reference surface of such shells is formed by the revolution of an arbitrary curve given on a plane by a discrete number of points that have coordinates with random errors of measurement (Grigolyuk and Kulikov, 1982).

Early tire models based on elementary structural analyses are discussed in survey papers by Frank and Hofferberth (1967), and Ridha (1980). So, it is not necessary to consider them here in detail. The modern tire computational model on the basis of a composite shell theory was first elaborated by Brewer (1973). He modeled tires by applying the geometrically non-linear Kirchhoff–Love theory of multilayered orthotropic shells. Qualitatively new tire computational models based on the geometrically non-linear first-order Timoshenko–Mindlin-type theory of multilayered anisotropic shells were developed by Grigolyuk and Kulikov (1981, 1988b, 1993), Kulikov (1996), Kulikov et al. (2000), and Noor et al. (1987). However, in these studies the transverse normal deformation response is not included.

Herein, the refined first-order global approximation theory of initially stressed multilayered anisotropic shells is developed. In global approximation theories, global through-the-thickness displacement, strain or stress approximations are introduced and as a result the multilayered shell is replaced by an equivalent single-layer shell (Grigolyuk and Kulikov, 1988b; Noor and Burton, 1990). Consequently, the order of the governing equations is independent on the number of layers of the shell. The simplest examples of these theories are the so-called first-order shear deformation theories based on the kinematic Timoshenko hypothesis (the linear distribution of displacements in the thickness direction).

The direct use of the traditional first-order global approximation theories for solving a series of important shell problems such as the contact problems is not always convenient. In these problems it is more convenient to select as unknown functions the tangential and transverse displacements of the face surfaces of the shell, since with the help of these displacements the kinematic requirement of no penetration of the contact bodies can be fulfilled. Furthermore, the proposed approach strongly simplifies a formulation of the non-linear strain–displacement relationships.

This theory is based on the refined kinematic Timoshenko hypothesis adopted for the displacement vector. The governing equations of the theory of initially stressed multilayered anisotropic shells are obtained by using the principle of the virtual work and partially non-linear Novozhilov's strain–displacement relationships. An outcome of this approach is that the equilibrium equations of the geometrically non-linear elasticity theory are satisfied pointwise into the shell with an exactitude acceptable for the thin shell structures.

On the basis of the proposed first-order global approximation theory, two computational models for solving the axisymmetric and non-axisymmetric problems of initially stressed multilayered anisotropic shells of revolution have been elaborated. The material of each layer of the shell is assumed to be linearly elastic, anisotropic, homogeneous or fiber reinforced, such that in each point there is a single surface of elastic symmetry parallel to the reference surface. The axisymmetric computational model is based on the Newton–Raphson method and the incremental method. In the case of the non-axisymmetric deformation of prestressed multilayered anisotropic shells of revolution the solution of the problem is carried out in two steps. First, the geometrically non-linear problem of the axisymmetric shell deformation on the basis of the first computational model is solved. Then, the geometrically linear problem for a prestressed shell of revolution subjected to non-axisymmetric loads is solved. For this purpose, the unknown functions and the external loads are expanded in the Fourier series in the circumferential coordinate.

The several numerical examples are presented. These examples include some relatively simple problems. Namely, the non-linear axisymmetric response of a cross-ply toroidal shell made of cord–rubber materials and subjected to inflation pressure; and the non-axisymmetric response of this shell subjected to inflation pressure, and localized loading acting on the outer surface.

2. Elasticity theory of initially stressed multilayered shells

Let us consider the shell built-up in the general case by the arbitrary superposition across the wall thickness of N thin layers of uniform thickness h_k. The kth layer may be defined as a three-dimensional body of volume V_k bounded by two surfaces S_{k-1} and S_k, located at the distances δ_{k-1} and δ_k measured with respect to the reference surface S, and the edge boundary surface Ω_k that is perpendicular to the reference surface (Fig. 1). The full edge boundary surface $\Omega = \Omega_1 + \Omega_2 + \cdots + \Omega_N$ is generated by the normals to the reference surface along the bounding curve Γ (with the arc length s) of this surface. It is also assumed that the bounding surfaces S_{k-1} and S_k are continuous, sufficiently smooth and without any singularities. Let the reference surface be referred to an orthogonal curvilinear coordinate system α_1 and α_2, which coincides with

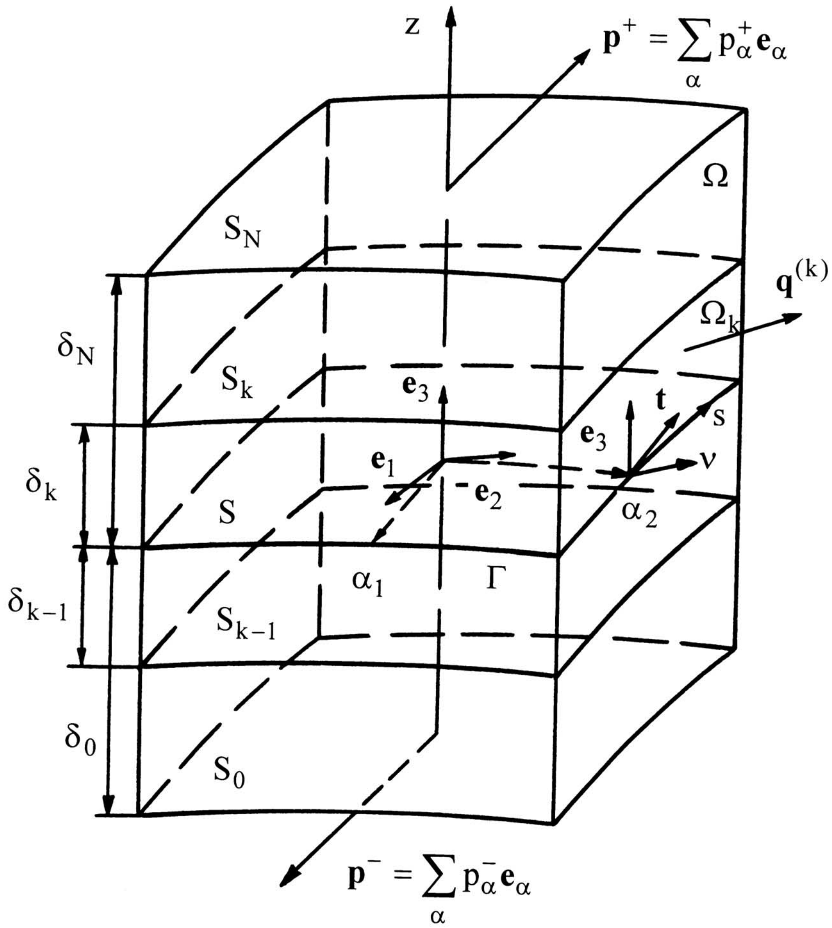

Fig. 1. Element of multilayered shell.

the lines of principal curvatures of its surface. The z axis is oriented along the outward unit vector $\mathbf{e}_3$ normal to the reference surface.

The constituent layers of the shell are supposed to be rigidly joined, so that no slip on contact surfaces and no separation of layers can occur. The material of each constituent layer is assumed to be linearly elastic, anisotropic, homogeneous or fiber reinforced, such that in each point there is a single surface of elastic symmetry parallel to the reference surface. Let $\widetilde{\sigma}_{\alpha\beta}^{(k)}$ be the initial stresses; $\widetilde{p}_\alpha^-$ and $\widetilde{p}_\alpha^+$ are the intensities of the initial external loading acting on the bottom surface S_0 and top surface S_N in the α_1, α_2 and z coordinate directions, respectively; $\widetilde{q}_v^{(k)}$, $\widetilde{q}_t^{(k)}$ and $\widetilde{q}_3^{(k)}$ are the intensities of the initial external loading acting on the edge boundary surface Ω_k in the v, t and z directions, where $\mathbf{v}$ and $\mathbf{t}$ are the normal and tangential unit vectors to the bounding curve Γ (Fig. 1). Here, and in the following developments, the index k identifies the belonging of any quantity to the kth layer ($k = \overline{1,N}$) and indices α and β take the values 1, 2 and 3; and indices i and j take the values 1 and 2.

Since the initial surface loads $\widetilde{p}_\alpha^-$, $\widetilde{p}_\alpha^+$, $\widetilde{q}_v^{(k)}$, $\widetilde{q}_t^{(k)}$, $\widetilde{q}_3^{(k)}$ and initial stresses $\widetilde{\sigma}_{\alpha\beta}^{(k)}$ constitute the self-equilibrated system and assuming the case of thin shells, we have

- the equilibrium equations of the three-dimensional elasticity theory for the kth layer:

$$\begin{aligned}
&\frac{1}{A_i}\frac{\partial\widetilde{\sigma}_{ii}^{(k)}}{\partial\alpha_i} + \frac{1}{A_j}\frac{\partial\widetilde{\sigma}_{ij}^{(k)}}{\partial\alpha_j} + \frac{\partial\widetilde{\sigma}_{i3}^{(k)}}{\partial z} + B_i\left(\widetilde{\sigma}_{ii}^{(k)} - \widetilde{\sigma}_{jj}^{(k)}\right) + 2B_j\widetilde{\sigma}_{ij}^{(k)} + k_i\widetilde{\sigma}_{i3}^{(k)} = 0, \quad (i \neq j),\\
&\frac{1}{A_1}\frac{\partial\widetilde{\sigma}_{13}^{(k)}}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial\widetilde{\sigma}_{23}^{(k)}}{\partial\alpha_2} + \frac{\partial\widetilde{\sigma}_{33}^{(k)}}{\partial z} + B_1\widetilde{\sigma}_{13}^{(k)} + B_2\widetilde{\sigma}_{23}^{(k)} - k_1\widetilde{\sigma}_{11}^{(k)} - k_2\widetilde{\sigma}_{22}^{(k)} = 0,
\end{aligned} \tag{1}$$

  where k_1 and k_2 are the principal curvatures of the reference surface; A_1 and A_2 are the Lamé coefficients of the reference surface; $B_i = (1/A_1A_2)(\partial A_j/\partial\alpha_i), \quad (i \neq j)$;
- the boundary conditions for the transverse stresses on the top surface S_N:

$$\widetilde{\sigma}_{\alpha 3}^{(N)} = \widetilde{p}_\alpha^+, \tag{2}$$

- the boundary conditions for the transverse stresses on the bottom surface S_0:

$$\widetilde{\sigma}_{\alpha 3}^{(1)} = \widetilde{p}_\alpha^-, \tag{3}$$

- the equilibrium conditions for the transverse stresses at the layer interfaces S_n:

$$\widetilde{\sigma}_{\alpha 3}^{(n+1)} = \widetilde{\sigma}_{\alpha 3}^{(n)} = \widetilde{\tau}_\alpha^{(n)}, \quad (n = \overline{1, N-1}), \tag{4}$$

  where $\widetilde{\tau}_\alpha$ are the initial interlaminar stresses;
- the boundary conditions on the edge boundary surfaces Ω_k:

$$\widetilde{\sigma}_{vv}^{(k)} = \widetilde{q}_v^{(k)}, \qquad \widetilde{\sigma}_{vt}^{(k)} = \widetilde{q}_t^{(k)}, \qquad \widetilde{\sigma}_{v3}^{(k)} = \widetilde{q}_3^{(k)}, \tag{5}$$

  where $\widetilde{\sigma}_{vv}^{(k)}$, $\widetilde{\sigma}_{vt}^{(k)}$, $\widetilde{\sigma}_{v3}^{(k)}$ are the components of the initial stress tensor of the kth layer in the coordinate system v, t, z.

The boundary value problem for the prestressed multilayered shell is defined by setting the additional loading p_α^-, p_α^+, $q_v^{(k)}$, $q_t^{(k)}$, $q_3^{(k)}$ (Fig. 1). As a result of this loading, the resulting stress state can be represented as

$$\sigma_{\alpha\beta}^{(k)r} = \widetilde{\sigma}_{\alpha\beta}^{(k)} + \sigma_{\alpha\beta}^{(k)}, \tag{6}$$

where $\sigma_{\alpha\beta}^{(k)}$ are the additional stresses of the kth layer.

The principle of the virtual work for the prestressed multilayered thin shell can be written in the following form (Washizu, 1982):

$$\sum_{k=1}^{N}\int\int_{V_k}\int\sum_{\alpha\leqslant\beta}\sigma_{\alpha\beta}^{(k)r}\delta\varepsilon_{\alpha\beta}^{(k)}A_1A_2\,\mathrm{d}\alpha_1\mathrm{d}\alpha_2\,\mathrm{d}z-\int\int_{S_N}\sum_{\alpha}\left(\widetilde{p}_{\alpha}^{+}+p_{\alpha}^{+}\right)\delta u_{\alpha}^{(N)}\,\mathrm{d}S+\int\int_{S_0}\sum_{\alpha}\left(\widetilde{p}_{\alpha}^{-}+p_{\alpha}^{-}\right)\delta u_{\alpha}^{(1)}\mathrm{d}S$$
$$+\sum_{n=1}^{N-1}\int\int_{S_n}\sum_{\alpha}\left(\widetilde{\tau}_{\alpha}^{(n)}+\tau_{\alpha}^{(n)}\right)\left(\delta u_{\alpha}^{(n+1)}-\delta u_{\alpha}^{(n)}\right)\mathrm{d}S-\sum_{k=1}^{N}\int\int_{\Omega_k}\Big[\left(\widetilde{q}_{v}^{(k)}+q_{v}^{(k)}\right)\delta u_{v}^{(k)}+\left(\widetilde{q}_{t}^{(k)}+q_{t}^{(k)}\right)\delta u_{t}^{(k)}$$
$$+\left(\widetilde{q}_{3}^{(k)}+q_{3}^{(k)}\right)\delta u_{3}^{(k)}\Big]\mathrm{d}S=0, \tag{7}$$

where $u_{\alpha}^{(k)}$ are the components of the displacement vector of the kth layer in the coordinate system α_1, α_2, z that are referred from the reference surface S; $u_{v}^{(k)}$, $u_{t}^{(k)}$ and $u_{3}^{(k)}$ are the components of the displacement vector of the kth layer in the coordinate system v, t, z; $\tau_{\alpha}^{(n)}$ are the interlaminar transverse stresses acting on the layer interfaces S_n, where $n=\overline{1,N-1}$.

The three-dimensional partially non-linear Novozhilov's strain–displacement relationships in the Lagrange description for the multilayered thin shell will be (Grigolyuk and Kulikov, 1988b)

$$\begin{aligned}
\varepsilon_{ii}^{(k)}&=\frac{1}{A_i}\frac{\partial u_i^{(k)}}{\partial\alpha_i}+B_ju_j^{(k)}+k_iu_3^{(k)}+\frac{1}{2}\left(\Theta_i^{(k)}\right)^2,\quad(i\neq j)\\
\varepsilon_{12}^{(k)}&=\frac{1}{A_1}\frac{\partial u_2^{(k)}}{\partial\alpha_1}+\frac{1}{A_2}\frac{\partial u_1^{(k)}}{\partial\alpha_2}-B_2u_1^{(k)}-B_1u_2^{(k)}+\Theta_1^{(k)}\Theta_2^{(k)},\\
\varepsilon_{i3}^{(k)}&=\frac{\partial u_i^{(k)}}{\partial z}+\Theta_i^{(k)},\qquad\varepsilon_{33}^{(k)}=\frac{\partial u_3^{(k)}}{\partial z}+\frac{1}{2}\left(\frac{\partial u_1^{(k)}}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial u_2^{(k)}}{\partial z}\right)^2,\\
\Theta_i^{(k)}&=\frac{1}{A_i}\frac{\partial u_3^{(k)}}{\partial\alpha_i}-k_iu_i^{(k)}.
\end{aligned}\tag{8}$$

In the expressions of tangential strains (8) only those non-linear geometrical terms that depend on $\Theta_1^{(k)}$ and $\Theta_2^{(k)}$ are retained. The remaining non-linear terms are discarded.

The governing equations of the geometrically non-linear elasticity theory for the prestressed multilayered thin shell can be derived by applying the principle of the virtual work (7). Substituting the strain–displacement relationships (8) into Eq. (7), using Gauss' theorem and taking into account Eqs. (1)–(6), one obtains the following variational equation:

$$\sum_{k=1}^{N}\int\int_{V_k}\int\sum_{\alpha}L_{\alpha}^{(k)}\delta u_{\alpha}^{(k)}A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}z-\int\int_{S_N}\sum_{\alpha}\left(s_{\alpha3}^{(N)}-p_{\alpha}^{+}\right)\delta u_{\alpha}^{(N)}\,\mathrm{d}S+\int\int_{S_0}\sum_{\alpha}\left(s_{\alpha3}^{(1)}-p_{\alpha}^{-}\right)\delta u_{\alpha}^{(1)}\,\mathrm{d}S$$
$$+\sum_{n=1}^{N-1}\int\int_{S_n}\sum_{\alpha}\Big[\left(s_{\alpha3}^{(n+1)}-\tau_{\alpha}^{(n)}\right)\delta u_{\alpha}^{(n+1)}-\left(s_{\alpha3}^{(n)}-\tau_{\alpha}^{(n)}\right)\delta u_{\alpha}^{(n)}\Big]\mathrm{d}S$$
$$-\sum_{k=1}^{N}\int\int_{\Omega_k}\Big[\left(\sigma_{vv}^{(k)}-q_{v}^{(k)}\right)\delta u_{v}^{(k)}+\left(\sigma_{vt}^{(k)}-q_{t}^{(k)}\right)\delta u_{t}^{(k)}+\left(\Sigma_{v3}^{(k)}-q_{3}^{(k)}\right)\delta u_{3}^{(k)}\Big]\mathrm{d}S=0, \tag{9}$$

where $L_{\alpha}^{(k)}$ are the three-dimensional non-linear differential operators, corresponding to Novozhilov's strain–displacement relationships (8), which can be written as follows:

$$
\begin{aligned}
L_i^{(k)} &= \frac{1}{A_i}\frac{\partial \sigma_{ii}^{(k)}}{\partial \alpha_i} + \frac{1}{A_j}\frac{\partial \sigma_{ij}^{(k)}}{\partial \alpha_j} + \frac{\partial s_{i3}^{(k)}}{\partial z} + B_i\left(\sigma_{ii}^{(k)} - \sigma_{jj}^{(k)}\right) + 2B_j\sigma_{ij}^{(k)} + k_i\Sigma_{i3}^{(k)}, \quad (i \neq j), \\
L_3^{(k)} &= \frac{1}{A_1}\frac{\partial \Sigma_{13}^{(k)}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial \Sigma_{23}^{(k)}}{\partial \alpha_2} + \frac{\partial s_{33}^{(k)}}{\partial z} + B_1\Sigma_{13}^{(k)} + B_2\Sigma_{23}^{(k)} - k_1\sigma_{11}^{(k)} - k_2\sigma_{22}^{(k)}, \\
\Sigma_{i3}^{(k)} &= \sigma_{i3}^{(k)} + \Theta_1^{(k)}\left(\widetilde{\sigma}_{1i}^{(k)} + \sigma_{1i}^{(k)}\right) + \Theta_2^{(k)}\left(\widetilde{\sigma}_{i2}^{(k)} + \sigma_{i2}^{(k)}\right), \\
s_{i3}^{(k)} &= \sigma_{i3}^{(k)} + \frac{\partial u_i^{(k)}}{\partial z}\left(\widetilde{\sigma}_{33}^{(k)} + \sigma_{33}^{(k)}\right), \qquad s_{33}^{(k)} = \sigma_{33}^{(k)},
\end{aligned}
\tag{10}
$$

where $\Sigma_{i3}^{(k)}$ and $s_{i3}^{(k)}$ are the generalized transverse shear stresses.

Equating the coefficients of $\delta u_\alpha^{(k)}$ to zero, one can obtain the fundamental equations of the geometrically non-linear elasticity theory of prestressed multilayered thin shells:

- the equilibrium equations for the kth layer:

$$L_\alpha^{(k)} = 0, \tag{11}$$

- the boundary conditions for the generalized transverse stresses on the top surface S_N:

$$s_{\alpha 3}^{(N)} = p_\alpha^+, \tag{12}$$

- the boundary conditions for the generalized transverse stresses on the bottom surface S_0:

$$s_{\alpha 3}^{(1)} = p_\alpha^-, \tag{13}$$

- the equilibrium conditions for the generalized transverse stresses at the layer interfaces S_n:

$$s_{\alpha 3}^{(n+1)} = s_{\alpha 3}^{(n)} = \tau_\alpha^{(n)}, \quad (n = \overline{1, N-1}), \tag{14}$$

- the boundary conditions on the edge boundary surfaces Ω_k:

$$\sigma_{\nu\nu}^{(k)} = q_\nu^{(k)}, \qquad \sigma_{\nu t}^{(k)} = q_t^{(k)}, \qquad \Sigma_{\nu 3}^{(k)} = q_3^{(k)}. \tag{15}$$

Additionally, we should invoke the generalized Hooke's law:

$$
\begin{aligned}
\sigma_{11}^{(k)} &= C_{11}^{(k)}\varepsilon_{11}^{(k)} + C_{12}^{(k)}\varepsilon_{22}^{(k)} + \underline{C_{13}^{(k)}\varepsilon_{33}^{(k)}} + C_{16}^{(k)}\varepsilon_{12}^{(k)}, \\
\sigma_{22}^{(k)} &= C_{12}^{(k)}\varepsilon_{11}^{(k)} + C_{22}^{(k)}\varepsilon_{22}^{(k)} + \underline{C_{23}^{(k)}\varepsilon_{33}^{(k)}} + C_{26}^{(k)}\varepsilon_{12}^{(k)}, \\
\sigma_{33}^{(k)} &= C_{13}^{(k)}\varepsilon_{11}^{(k)} + C_{23}^{(k)}\varepsilon_{22}^{(k)} + C_{33}^{(k)}\varepsilon_{33}^{(k)} + C_{36}^{(k)}\varepsilon_{12}^{(k)}, \\
\sigma_{12}^{(k)} &= C_{16}^{(k)}\varepsilon_{11}^{(k)} + C_{26}^{(k)}\varepsilon_{22}^{(k)} + \underline{C_{36}^{(k)}\varepsilon_{33}^{(k)}} + C_{66}^{(k)}\varepsilon_{12}^{(k)}, \\
\sigma_{23}^{(k)} &= C_{44}^{(k)}\varepsilon_{23}^{(k)} + C_{45}^{(k)}\varepsilon_{13}^{(k)}, \\
\sigma_{13}^{(k)} &= C_{45}^{(k)}\varepsilon_{23}^{(k)} + C_{55}^{(k)}\varepsilon_{13}^{(k)},
\end{aligned}
\tag{16}
$$

where $C_{\ell m}^{(k)}$ are the stiffness coefficients of the kth layer ($\ell, m = \overline{1, 6}$).

So, we have all the fundamental equations (6), (8), (11)–(16) for finding the resulting stress state of the prestressed multilayered anisotropic thin shell.

3. First-order theory of initially stressed multilayered shells

The first-order global approximation theory of multilayered shells is based on the linear approximation for the displacement vector in the thickness direction

$$\begin{aligned} & u_\alpha^{(k)} = N^-(z)v_\alpha^- + N^+(z)v_\alpha^+, \\ & N^-(z) = (\delta_N - z)/h, \qquad N^+(z) = (z-\delta_0)/h, \end{aligned} \tag{17}$$

where $v_\alpha^-(\alpha_1,\alpha_2)$ and $v_\alpha^+(\alpha_1,\alpha_2)$ are the tangential and transverse displacements of the bottom surface S_0 and top surface S_N; $N^-(z)$ and $N^+(z)$ are the linear shape functions. The linear approximation (17) may be considered as a refined Timoshenko hypothesis (for example, works by Grigolyuk and Kulikov (1988b) or by Noor and Burton (1990), where as unknown functions, the displacements of the reference surface and rotation components are selected). The advantage of the proposed approach is obvious, since with the help of the displacements v_α^- and v_α^+, the kinematic boundary conditions on the face surfaces of the shell, and in particular, the conditions of no penetration of the contact bodies can be formulated. Besides, this provides a convenient way to express the non-linear strain–displacement relationships in terms of face surface strains (Eqs. (21) and (22)).

Substituting the displacements from Eq. (17) into the strain–displacement relationships (8) and variational equation (9), and taking into account that a shell is thin, the following equations of the geometrically non-linear theory of prestressed multilayered shells are obtained:

- the equilibrium equations of the three-dimensional elasticity theory for the kth layer (10) and (11), where we should set

$$\begin{aligned} & s_{i3}^{(k)} = \sigma_{i3}^{(k)} + \beta_i\left(\widetilde{\sigma}_{33}^{(k)} + \sigma_{33}^{(k)}\right), \qquad \Theta_i^{(k)} = -N^-(z)\theta_i^- - N^+(z)\theta_i^+, \\ & \beta_i = \frac{1}{h}\left(v_i^+ - v_i^-\right), \quad \theta_i^\pm = k_i v_i^\pm - \frac{1}{A_i}\frac{\partial v_3^\pm}{\partial \alpha_i}, \end{aligned} \tag{18}$$

- the boundary conditions for the generalized transverse stresses on the top surface (12);
- the boundary conditions for the generalized transverse stresses on the bottom surface (13);
- the equilibrium conditions for the generalized transverse stresses at the layer interfaces (14);
- the natural boundary conditions on the edge boundary surface Ω:

$$\left(H_{vv}^\pm - \hat{H}_{vv}^\pm\right)\delta v_v^\pm = 0, \qquad \left(H_{vt}^\pm - \hat{H}_{vt}^\pm\right)\delta v_t^\pm = 0, \qquad \left(S_{v3}^\pm - \hat{H}_{v3}^\pm\right)\delta v_3^\pm = 0, \tag{19}$$

where $H_{vv}^\pm$, $H_{vt}^\pm$ and $S_{v3}^\pm$ are the generalized stress resultants:

$$\begin{aligned} & H_{vv}^\pm = \sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\sigma_{vv}^{(k)}N^\pm(z)\mathrm{d}z, \qquad H_{vt}^\pm = \sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\sigma_{vt}^{(k)}N^\pm(z)\mathrm{d}z, \\ & S_{v3}^\pm = \sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\Sigma_{v3}^{(k)}N^\pm(z)\mathrm{d}z \end{aligned} \tag{20}$$

and $\hat{H}_{vv}^\pm$, $\hat{H}_{vt}^\pm$, $\hat{H}_{v3}^\pm$ are the generalized loading resultants that are obtained from Eq. (20) by replacing the stresses $\sigma_{vv}^{(k)}$, $\sigma_{vt}^{(k)}$, $\Sigma_{v3}^{(k)}$ by intensities of the external loads $q_v^{(k)}$, $q_t^{(k)}$, $q_3^{(k)}$ acting in the v, t, z directions, correspondingly;
- the strain–displacement relationships

$$\begin{aligned} & \varepsilon_{ij}^{(k)} = N^-(z)E_{ij}^- + N^+(z)E_{ij}^+, \qquad \varepsilon_{i3}^{(k)} = N^-(z)E_{i3}^- + N^+(z)E_{i3}^+, \\ & \varepsilon_{33}^{(k)} = E_{33} = \beta_3 + \frac{1}{2}\beta_1^2 + \frac{1}{2}\beta_2^2, \end{aligned} \tag{21}$$

where E_{ij}^-, E_{i3}^- and E_{ij}^+, E_{i3}^+ are the tangential and transverse shear strains of the bottom and top surfaces of the shell, respectively:

$$
\begin{aligned}
&E_{ii}^{\pm} = e_{ii}^{\pm} + \frac{1}{2}\left(\theta_i^{\pm}\right)^2, \qquad E_{12}^{\pm} = e_{12}^{\pm} + \theta_1^{\pm}\theta_2^{\pm}, \\
&e_{ii}^{\pm} = \frac{1}{A_i}\frac{\partial v_i^{\pm}}{\partial \alpha_i} + B_j v_j^{\pm} + k_i v_3^{\pm} \quad (i \neq j), \qquad e_{12}^{\pm} = \frac{1}{A_1}\frac{\partial v_2^{\pm}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial v_1^{\pm}}{\partial \alpha_2} - B_2 v_1^{\pm} - B_1 v_2^{\pm}, \\
&E_{i3}^{\pm} = \beta_i - \theta_i^{\pm}, \qquad \beta_\alpha = \frac{1}{h}\left(v_\alpha^{+} - v_\alpha^{-}\right), \qquad \theta_i^{\pm} = k_i v_i^{\pm} - \frac{1}{A_i}\frac{\partial v_3}{\partial \alpha_i}.
\end{aligned}
\tag{22}
$$

Note that the tangential strains $\varepsilon_{ij}^{(k)}$ are distributed over the shell thickness according to the linear law, since the refined Timoshenko hypothesis has been adopted. As can be seen from Fig. 2, it is an acceptable assumption for the thin shell structures. Really, better expressions for the tangential strains can be written by using the quadratic approximation that are exact for the proposed non-linear shell theory, i.e.

$$
\varepsilon_{ii}^{(k)e} = N^-(z)e_{ii}^- + N^+(z)e_{ii}^+ + \frac{1}{2}\left(\Theta_i^{(k)}\right)^2, \qquad \varepsilon_{12}^{(k)e} = N^-(z)e_{12}^- + N^+(z)e_{12}^+ + \Theta_1^{(k)}\Theta_2^{(k)},
$$

where the functions $\Theta_i^{(k)}$ are defined by formulas (18). It is apparent that from the previous equations and Eqs. (21) and (22) follow that the coupling conditions $\varepsilon_{ij}^{(1)e}(\delta_0) = \varepsilon_{ij}^{(1)}(\delta_0) = E_{ij}^-$ and $\varepsilon_{ij}^{(N)e}(\delta_N) = \varepsilon_{ij}^{(N)}(\delta_N) = E_{ij}^+$ are satisfied; and the values of these strains will always coincide for the geometrically linear shell theory.

Multiplying the equations of the three-dimensional elasticity theory (11) by shape functions $N^-(z)$, $N^+(z)$, and integrating them across the shell thickness with account of the boundary conditions (12), (13) and equilibrium conditions (14), six non-linear equilibrium equations of the initially stressed multilayered thin shell in terms of stress resultants are obtained:

$$
\begin{aligned}
&\frac{1}{A_i}\frac{\partial H_{ii}^{\pm}}{\partial \alpha_i} + \frac{1}{A_j}\frac{\partial H_{ij}^{\pm}}{\partial \alpha_j} + B_i\left(H_{ii}^{\pm} - H_{jj}^{\pm}\right) + 2B_j H_{ij}^{\pm} + k_i S_{i3}^{\pm} \mp \frac{1}{h}P_{i3} \pm p_i^{\pm} = 0, \quad (i \neq j), \\
&\frac{1}{A_1}\frac{\partial S_{13}^{\pm}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial S_{23}^{\pm}}{\partial \alpha_2} + B_1 S_{13}^{\pm} + B_2 S_{23}^{\pm} - k_1 H_{11}^{\pm} - k_2 H_{22}^{\pm} \mp \frac{1}{h}T_{33} \pm p_3^{\pm} = 0,
\end{aligned}
\tag{23}
$$

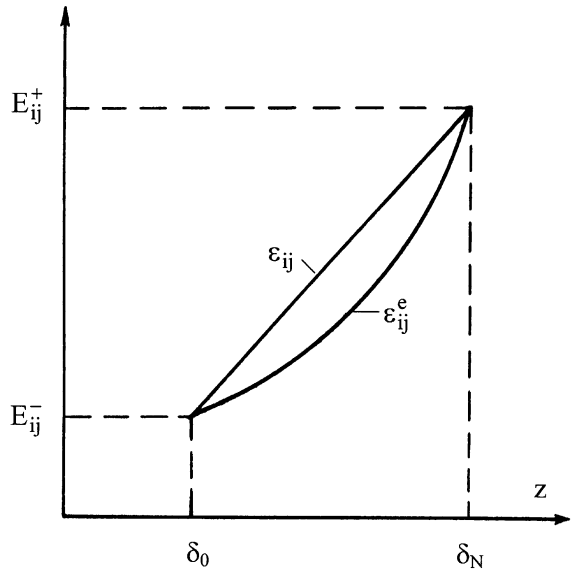

Fig. 2. Distribution of tangential strains ε_{ij}^e and ε_{ij} over shell thickness.

where $H^{\pm}_{i\alpha}$, $S^{\pm}_{i3}$ and P_{i3} are the generalized stress resultants, and $T_{\alpha 3}$ are the classical stress resultants:

$$\begin{aligned} H^{\pm}_{i\alpha} &= \sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k} \sigma^{(k)}_{i\alpha} N^{\pm}(z)\mathrm{d}z, \qquad S^{\pm}_{i3} = \sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k} \Sigma^{(k)}_{i3} N^{\pm}(z)\mathrm{d}z, \\ P_{i3} &= \sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k} s^{(k)}_{i3}\mathrm{d}z, \qquad T_{\alpha 3} = \sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k} \sigma^{(k)}_{\alpha 3}\mathrm{d}z. \end{aligned} \tag{24}$$

In order to obtain the constitutive equations for the stress resultants, the equations of the generalized Hooke's law (16) should be used. Unfortunately, such approach cannot correctly describe the shells made of incompressible materials or nearly incompressible materials having Poisson's coefficients $\nu_{\alpha\beta} \approx 0.5(\alpha \neq \beta)$. To avoid this contradiction, we should simplify the equations of the generalized Hooke's law for the tangential stresses (16) omitting the underlined terms. It is an acceptable assumption for thin shell structures.

Indeed, consider the orthotropic layer of the shell whose axes of symmetry $\alpha^{(k)}_{1'}$, $\alpha^{(k)}_{2'}$ and z do not coincide with the coordinate directions α_1, α_2 and z. In axes of symmetry, the equations of the generalized Hooke's law will be

$$\varepsilon^{(k)}_{1'1'} = \frac{1}{E^{(k)}_1}\sigma^{(k)}_{1'1'} - \frac{\nu^{(k)}_{21}}{E^{(k)}_2}\sigma^{(k)}_{2'2'} - \frac{\nu^{(k)}_{31}}{E^{(k)}_3}\sigma^{(k)}_{33}, \qquad \varepsilon^{(k)}_{2'2'} = -\frac{\nu^{(k)}_{12}}{E^{(k)}_1}\sigma^{(k)}_{1'1'} + \frac{1}{E^{(k)}_2}\sigma^{(k)}_{2'2'} - \frac{\nu^{(k)}_{32}}{E^{(k)}_3}\sigma^{(k)}_{33}, \tag{25}$$

$$\varepsilon^{(k)}_{33} = -\frac{\nu^{(k)}_{13}}{E^{(k)}_1}\sigma^{(k)}_{1'1'} - \frac{\nu^{(k)}_{23}}{E^{(k)}_2}\sigma^{(k)}_{2'2'} + \frac{1}{E^{(k)}_3}\sigma^{(k)}_{33}, \tag{26}$$

$$\varepsilon^{(k)}_{2'3} = \frac{1}{G^{(k)}_{23}}\sigma^{(k)}_{2'3}, \qquad \varepsilon^{(k)}_{1'3} = \frac{1}{G^{(k)}_{13}}\sigma^{(k)}_{1'3}, \qquad \varepsilon^{(k)}_{1'2'} = \frac{1}{G^{(k)}_{12}}\sigma^{(k)}_{1'2'}, \tag{27}$$

where $E^{(k)}_1$, $E^{(k)}_2$ and $E^{(k)}_3$ are the elastic moduli in the $\alpha^{(k)}_{1'}$, $\alpha^{(k)}_{2'}$ and z directions; $G^{(k)}_{12}$, $G^{(k)}_{13}$ and $G^{(k)}_{23}$ are the shear moduli. From reasons of symmetry, we have $\nu^{(k)}_{\alpha\beta}/E^{(k)}_{\alpha} = \nu^{(k)}_{\beta\alpha}/E^{(k)}_{\beta}$ $(\alpha \neq \beta)$.

As a shell is thin, with an exactitude acceptable to engineering calculations it is possible to accept the following assumption: $\sigma^{(k)}_{33} \ll \sigma^{(k)}_{1'1'}, \sigma^{(k)}_{2'2'}$. Neglecting the transverse normal stress in Eq. (25), and solving for the tangential stresses, we find

$$\begin{aligned} &\sigma^{(k)}_{1'1'} = Q^{(k)}_{11}\varepsilon^{(k)}_{1'1'} + Q^{(k)}_{12}\varepsilon^{(k)}_{2'2'}, \qquad \sigma^{(k)}_{2'2'} = Q^{(k)}_{12}\varepsilon^{(k)}_{1'1'} + Q^{(k)}_{22}\varepsilon^{(k)}_{2'2'}, \\ &Q^{(k)}_{11} = \frac{E^{(k)}_1}{1-\nu^{(k)}_{12}\nu^{(k)}_{21}}, \qquad Q^{(k)}_{22} = \frac{E^{(k)}_2}{1-\nu^{(k)}_{12}\nu^{(k)}_{21}}, \qquad Q^{(k)}_{12} = \frac{\nu^{(k)}_{12}E^{(k)}_2}{1-\nu^{(k)}_{12}\nu^{(k)}_{21}}. \end{aligned} \tag{28}$$

Substituting the tangential stresses $\sigma^{(k)}_{1'1'}$ and $\sigma^{(k)}_{2'2'}$ into Eq. (26), and solving for the transverse normal stress, we obtain

$$\begin{aligned} &\sigma^{(k)}_{33} = Q^{(k)}_{13}\varepsilon^{(k)}_{1'1'} + Q^{(k)}_{23}\varepsilon^{(k)}_{2'2'} + Q^{(k)}_{33}\varepsilon^{(k)}_{33}, \\ &Q^{(k)}_{13} = \frac{\nu^{(k)}_{13} + \nu^{(k)}_{12}\nu^{(k)}_{23}}{1-\nu^{(k)}_{12}\nu^{(k)}_{21}}E^{(k)}_3, \qquad Q^{(k)}_{23} = \frac{\nu^{(k)}_{23} + \nu^{(k)}_{21}\nu^{(k)}_{13}}{1-\nu^{(k)}_{12}\nu^{(k)}_{21}}E^{(k)}_3, \qquad Q^{(k)}_{33} = E^{(k)}_3. \end{aligned} \tag{29}$$

In coordinate directions α_1, α_2 and z, the equations of the generalized Hooke's law (27)–(29) can be represented in the following form:

$$\begin{aligned}
\sigma_{11}^{(k)} &= C_{11}^{(k)}\varepsilon_{11}^{(k)} + C_{12}^{(k)}\varepsilon_{22}^{(k)} + C_{16}^{(k)}\varepsilon_{12}^{(k)},\\
\sigma_{22}^{(k)} &= C_{12}^{(k)}\varepsilon_{11}^{(k)} + C_{22}^{(k)}\varepsilon_{22}^{(k)} + C_{26}^{(k)}\varepsilon_{12}^{(k)},\\
\sigma_{12}^{(k)} &= C_{16}^{(k)}\varepsilon_{11}^{(k)} + C_{26}^{(k)}\varepsilon_{22}^{(k)} + C_{66}^{(k)}\varepsilon_{12}^{(k)}.
\end{aligned} \tag{30}$$

The remaining equations of the generalized Hooke's law are given by formulas (16). The components of the stiffness matrix $C_{\ell m}^{(k)}$ in new axes can be found, for example, in paper by Kulikov and Plotnikova (1999).

It is a well-known fact that in the Timoshenko–Mindlin-type shell theory the equations of the generalized Hooke's law for the transverse shear and normal stresses are not satisfied pointwise, but can be satisfied in an integral sense. Therefore, according to formulas (16) and (24), the following integral equations must be fulfilled:

$$\begin{aligned}
&\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\left(\sigma_{33}^{(k)} - C_{13}^{(k)}\varepsilon_{11}^{(k)} - C_{23}^{(k)}\varepsilon_{22}^{(k)} - C_{33}^{(k)}\varepsilon_{33}^{(k)} - C_{36}^{(k)}\varepsilon_{12}^{(k)}\right)\mathrm{d}z = 0,\\
&\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\left(\sigma_{23}^{(k)} - C_{44}^{(k)}\varepsilon_{23}^{(k)} - C_{45}^{(k)}\varepsilon_{13}^{(k)}\right)N^{\pm}(z)\mathrm{d}z = 0,\\
&\sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k}\left(\sigma_{13}^{(k)} - C_{45}^{(k)}\varepsilon_{23}^{(k)} - C_{55}^{(k)}\varepsilon_{13}^{(k)}\right)N^{\pm}(z)\mathrm{d}z = 0.
\end{aligned} \tag{31}$$

With the help of the constitutive Eqs. (30) and (31), strain–displacement relationships (21) and formulas (24), we can obtain the constitutive equations for the stress resultants of the non-linear first-order global approximation theory of initially stressed multilayered anisotropic shells. However, due to their intricacy, these will not be displayed here.

Now, we have an opportunity to satisfy pointwise the equations of the three-dimensional elasticity theory (11) exactly for a plate and approximately for a shell with an exactitude acceptable for thin shell structures. Integrating the Eq. (11) across the shell thickness from δ_0 to z and using the boundary conditions on the bottom surface (13) and equilibrium conditions at the layer interfaces (14), one can obtain the expressions for the generalized transverse stresses

$$\begin{aligned}
s_{i3}^{(k)} &= p_i^- - \frac{1}{A_i}\frac{\partial \Lambda_{ii}^{(k)}}{\partial \alpha_i} - \frac{1}{A_j}\frac{\partial \Lambda_{ij}^{(k)}}{\partial \alpha_j} - B_i\left(\Lambda_{ii}^{(k)} - \Lambda_{jj}^{(k)}\right) - 2B_j\Lambda_{ij}^{(k)} - k_iR_{i3}^{(k)}, \quad (i \neq j),\\
s_{33}^{(k)} &= p_3^- - \frac{1}{A_1}\frac{\partial R_{13}^{(k)}}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial R_{23}^{(k)}}{\partial \alpha_2} - B_1R_{13}^{(k)} - B_2R_{23}^{(k)} + k_1\Lambda_{11}^{(k)} + k_2\Lambda_{22}^{(k)},
\end{aligned} \tag{32}$$

where $\Lambda_{ij}^{(k)}$ and $R_{i3}^{(k)}$ are the new stress resultants depending on the transverse coordinate:

$$\Lambda_{ij}^{(k)} = \sum_{n=1}^{k-1}\int_{\delta_{n-1}}^{\delta_n}\sigma_{ij}^{(n)}\,\mathrm{d}z + \int_{\delta_{k-1}}^{z}\sigma_{ij}^{(k)}\,\mathrm{d}z, \qquad R_{i3}^{(k)} = \sum_{n=1}^{k-1}\int_{\delta_{n-1}}^{\delta_n}\Sigma_{i3}^{(n)}\,\mathrm{d}z + \int_{\delta_{k-1}}^{z}\Sigma_{i3}^{(k)}\mathrm{d}z. \tag{33}$$

It is important to note that from Eqs. (23), (32) and (33) follow that the boundary conditions for the generalized transverse stresses on the top shell surface (12) and are also satisfied, since $\Lambda_{ij}^{(N)}(\delta_N) = H_{ij}^- + H_{ij}^+$ and $R_{i3}^{(N)}(\delta_N) = S_{i3}^- + S_{i3}^+$.

Finally, from formulas (18), we can find the transverse stresses

$$\sigma_{i3}^{(k)} = s_{i3}^{(k)} - \beta_i\left(\widetilde{\sigma}_{33}^{(k)} + s_{33}^{(k)}\right), \qquad \sigma_{33}^{(k)} = s_{33}^{(k)}. \tag{34}$$

So, all governing relationships of the refined non-linear first-order global approximation theory of prestressed multilayered anisotropic thin shells have been derived.

4. Axisymmetric deformation of initially stressed multilayered shells of revolution

Let us consider the prestressed multilayered anisotropic shell of revolution with uniform circumferential properties subjected to axisymmetric loading. It is assumed that the initial stresses $\widetilde{\sigma}_{\alpha\beta}^{(k)}$ are independent of the circumferential coordinate. In this case, the shell will deform axisymmetrically remaining as a body of revolution, and the displacements of the face shell surfaces $v_1^{\pm}$, $v_2^{\pm}$ and $v_3^{\pm}$ will depend only on the meridional coordinate s.

Let $\mathbf{Y}$ be the state vector given by

$$\mathbf{Y} = \left\| H_{11}^-, H_{11}^+, H_{12}^-, H_{12}^+, S_{13}^-, S_{13}^+, v_1^-, v_1^+, v_2^-, v_2^+, v_3^-, v_3^+ \right\|^{\mathrm{T}}, \tag{35}$$

where the superscript T denotes transposition. Taking into account the relationships (22), (23) and (35) and constitutive equations, we can write the governing system of non-linear differential equations in the following vector form:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}s} = \mathbf{F}(s, \mathbf{Y}). \tag{36}$$

The boundary conditions of the axisymmetric deformation problem according to formulas (19) can be written as follows:

$$\begin{aligned} &Y_n(s^-)\ell_n + Y_{n+6}(s^-)(1 - \ell_n) = 0, \\ &Y_n(s^+)\ell_{n+6} + Y_{n+6}(s^+)(1 - \ell_{n+6}) = 0, \end{aligned} \tag{37}$$

where Y_n and Y_{n+6} are the components of the state vector $\mathbf{Y}$; ℓ_n and ℓ_{n+6} are the boundary coefficients, which may take the values 0 and 1, and define any homogeneous static or kinematic boundary conditions at the left edge $s = s^-$ and right edge $s = s^+$ of the shell, where $n = \overline{1,6}$.

The non-linear boundary value problem (36) and (37) can be reduced to a sequence of linear boundary value problems by using the Newton–Raphson method as

$$\frac{\mathrm{d}\mathbf{Y}^{[m+1]}}{\mathrm{d}s} = \mathbf{A}\left(s, \mathbf{Y}^{[m]}\right) \cdot \mathbf{Y}^{[m+1]} + \mathbf{G}\left(s, \mathbf{Y}^{[m]}\right). \tag{38}$$

The linear boundary value problem (37) and (38) is solved by application of the discrete orthogonalization method used for solving the multilayered composite shells of revolution by Grigolyuk and Kulikov (1981). The process starts with $\mathbf{Y}^{[0]} = \mathbf{0}$, and we carry on it until the inequality

$$\max_{\ell} \left| \left(Y_\ell^{[m+1]} - Y_\ell^{[m]} \right) \Big/ Y_\ell^{[m+1]} \right| < \varepsilon$$

will be satisfied for a priori chosen parameter ε, where $\ell = \overline{1,12}$.

As a numerical example, we consider a linear response of a three-layered composite plate (Fig. 3). It is assumed that each layer possesses a single plane of elastic symmetry parallel to the middle plane. The plate is simply supported on the ends $s = 0$ and $s = \ell$, and is subjected to the normal loading $p_3^+ = q \sin \pi s/\ell$. Let us consider the class of problems known as a cylindrical bending, where all components of the displacement vector, and the strain and stress tensors are dependent only on the s and z coordinates. The material characteristics of each layer were taken to be those typical of a high modulus graphite–epoxy composite (Pagano, 1970): $E_{\mathrm{L}} = 25E$, $E_{\mathrm{T}} = E_{\mathrm{Z}} = E$, $G_{\mathrm{LT}} = G_{\mathrm{LZ}} = 0.5E$, $G_{\mathrm{TZ}} = 0.2E$, $\nu_{\mathrm{LT}} = \nu_{\mathrm{LZ}} = \nu_{\mathrm{TZ}} = 0.25$, where the subscripts L, T and Z refer to the longitudinal, transverse and thickness directions of the individual ply and $E = 6896$ MPa. Let the ply thicknesses and ply orientations, respectively, be $(h/4, h/2, h/4)$ and $(\gamma, -\gamma, \gamma)$, where γ is measured in the clockwise direction from s to the fiber direction. Fig. 4 shows the distribution of the transverse shear stresses in the thickness direction at the cross-section (at $s = 0$) for the ply angle $\gamma = 30°$, and the dimensionless parameter $\xi = 4$ and $\xi = 10$, where $\xi = \ell/h$. The solid curves

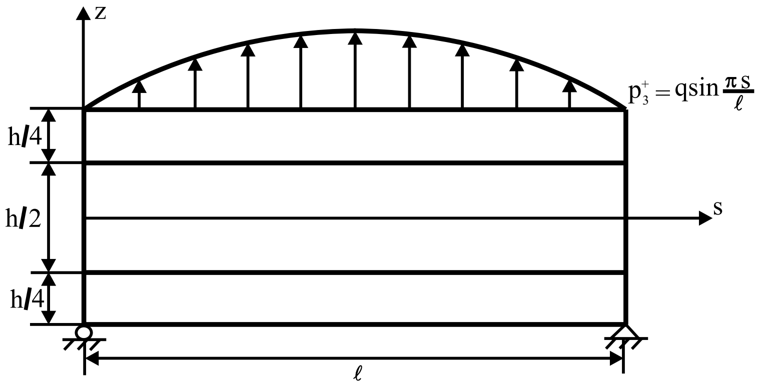

Fig. 3. Cylindrical bending of three-layered anisotropic plate.

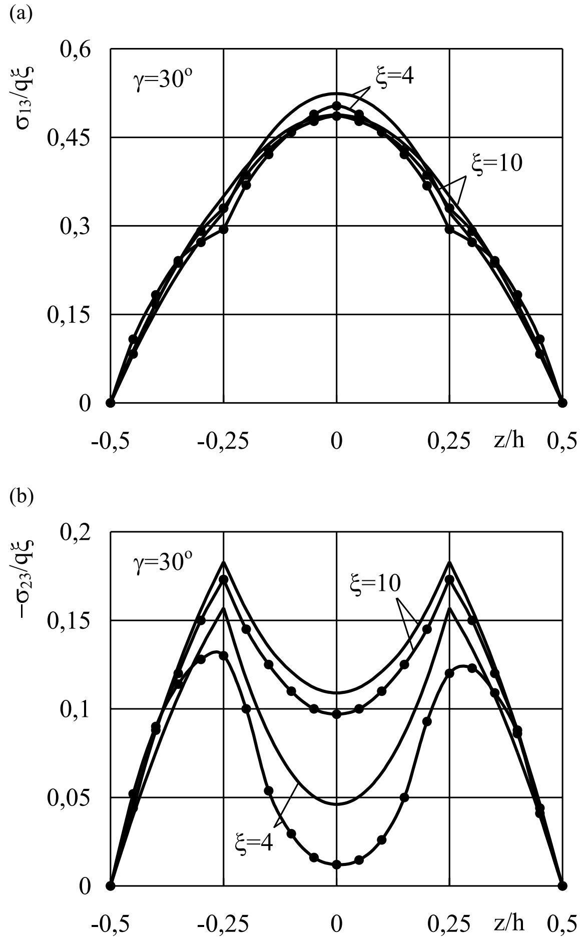

Fig. 4. Distribution of transverse shear stresses in thickness direction at cross-section ($s = 0$): (a) $\sigma_{13}/q\xi$; (b) $\sigma_{23}/q\xi$; () – present theory and (–•–•–) – Pagano (1970).

display the results obtained by using this theory, while Pagano's exact solution is denoted by curves marked by ●. It is seen that the proposed theory gives acceptable results for the moderately thick composite plates.

As the second numerical example, we consider a relatively simple problem of the non-linear axisymmetric response of the multilayered anisotropic tire. For the sake of simplicity, the tire is modeled as a four-layered anisotropic toroidal shell (so-called bias-ply tire), which has a circular cross-section (Fig. 5). The shell is subjected to uniform inflation pressure $p_3^- = -q$, where $q = 0.15$ MPa. The material characteristics of the layers are taken to be those typical of cord–rubber composites (Kulikov, 1996): $E_L = 510.45$ MPa, $E_T = E_Z = 6.91$ MPa, $G_{LT} = G_{LZ} = 2.33$ MPa, $G_{TZ} = 1.77$ MPa, $\nu_{LT} = \nu_{LZ} = 0.46$, $\nu_{TZ} = 0.95$. Let the geometrical characteristics of the inner surface of the shell are $R_1 = 50$ mm and $R_0 = 250$ mm; thicknesses of the shell and plies are $h = 4.8$ mm and $h_k = 1.2$ mm; ply orientations are $\gamma_k = (-1)^{k-1}\gamma$, where $\gamma = 52°$ and $k = \overline{1,4}$. The tire is assumed to be rigidly clamped at the rim (at $\psi = \pm 120°$).

This non-linear problem can be also solved by using the incremental method (Washizu, 1982). Let the tire be loaded to 0.15 MPa inflation pressure in five load steps, i.e., $q_n = 0.03n$ MPa, where $n = \overline{1,5}$. At each of the load steps, the geometrically linear problem for a prestressed shell of revolution is solved. It should be noted that the effects of the meridian stretching and thickness variation under the new geometry computation were not taken into account. Other feature of this approach is the non-conservative character of the pressure loading, since the displacements are referred at each of the load steps from a new reference surface.

The numerical results presented in Fig. 6 have been obtained by using the Newton–Raphson method (curves marked by ●) and the incremental method (the solid lines with various values of the load parameter

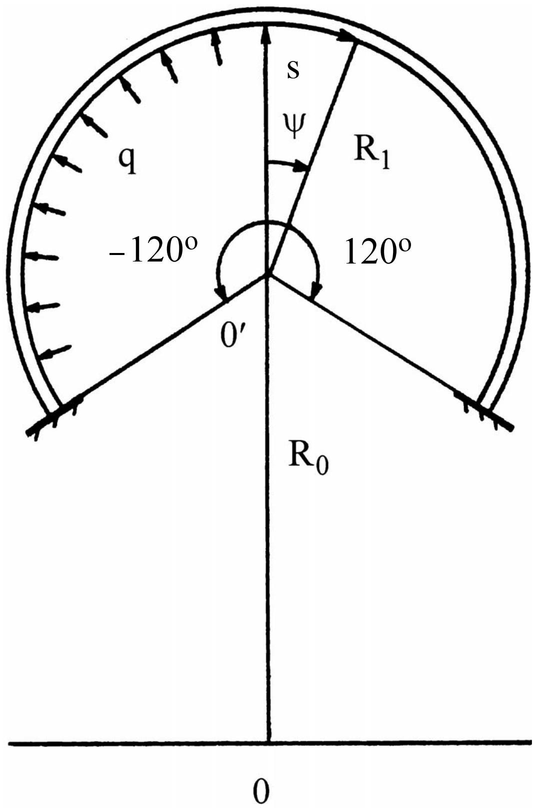

Fig. 5. Four-layered composite toroidal shell subjected to inflation pressure.

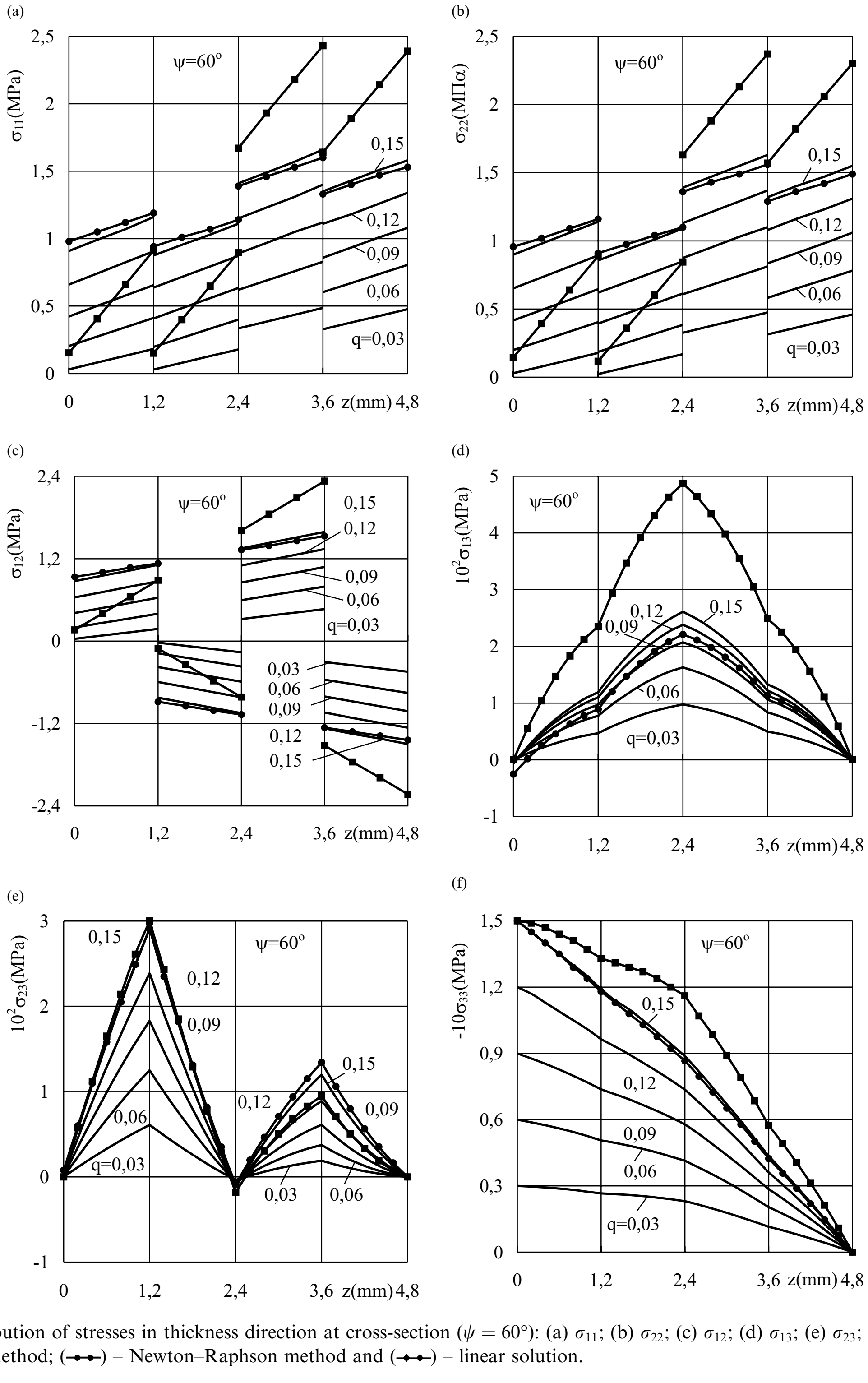

Fig. 6. Distribution of stresses in thickness direction at cross-section ($\psi = 60°$): (a) σ_{11}; (b) σ_{22}; (c) σ_{12}; (d) σ_{13}; (e) σ_{23}; (f) σ_{33}; (—) – incremental method; (–•–) – Newton–Raphson method and (–▪–) – linear solution.

q). Note that only four iterations were required for finding the solution of the non-linear problem with the given accuracy $\varepsilon = 10^{-4}$. Additionally, in Fig. 4, the solution of the geometrically linear problem is given (curves marked by ◆). The distribution of the stress components in the thickness direction is shown for the middle cross-section (at $\psi = 60°$). It is seen that both numerical solutions of the non-linear problem lead to similar results. Note that transverse shear stresses σ_{13} and σ_{23} obtained by using the Newton-Raphson method, and incremental method do not vanish at the inner surface of the shell and are discontinuous at the layer interfaces. It can be explained by allowing for the non-linear terms in formula (34). However, this effect is appreciable only for the finite deflection problems.

As already said, two layers of this cord–rubber composite are put together with $\pm\gamma$ fiber orientations with respect to the meridional direction. Each layer separately would try to exhibit the shear coupling behavior. Their shearing action would be in opposite directions due to their opposite fiber orientations. The mutual interaction between the layers would try to restrict in-plane shear motions and as a result would generate transverse shear stresses σ_{23} that are essential in pneumatic tires. It is apparent that in a case of using the traditional non-linear theory of laminated orthotropic shells we will lose this effect. [1] In this connection let us pay attention to the same order of the transverse shear stresses σ_{13} and σ_{23} that it is noticeable, namely, for the non-linear problem. It points to an essential influence of anisotropy and geometrical non-linearity on the stress field in cord–rubber toroidal shells.

5. Non-axisymmetric deformation of initially stressed multilayered shells of revolution

Consider the initially stressed multilayered anisotropic shell of revolution with uniform circumferential properties subjected to non-axisymmetric loads $p_1^\pm$, $p_2^\pm$ and $p_3^\pm$, acting on the bottom and top surfaces in the s, φ and z directions, correspondingly. As in the previous section, we will assume that the initial stresses, $\tilde{\sigma}_{\alpha\beta}^{(k)}$ are independent on the circumferential coordinate φ.

Let us suppose that the state vectors $\mathbf{X}$, $\mathbf{Z}$ and loading vectors $\mathbf{P}$, $\mathbf{Q}$ defined as

$$\begin{aligned}
\mathbf{X} &= \left\| H_{11}^-, H_{11}^+, H_{22}^-, H_{22}^+, H_{13}^-, H_{13}^+, S_{13}^-, S_{13}^+, P_{13}, T_{13}, v_1^-, v_1^+, v_3^-, v_3^+, E_{11}^-, E_{11}^+, E_{22}^-, E_{22}^+, E_{13}^-, E_{13}^+, E_{33}, \beta_1, \theta_1^-, \theta_1^+ \right\|^{\mathrm{T}}, \\
\mathbf{Z} &= \left\| H_{12}^-, H_{12}^+, H_{23}^-, H_{23}^+, S_{23}^-, S_{23}^+, P_{23}, T_{23}, v_2^-, v_2^+, E_{12}^-, E_{12}^+, E_{23}^-, E_{23}^+, \beta_2, \theta_2^-, \theta_2^+ \right\|^{\mathrm{T}} \\
\mathbf{P} &= \left\| p_1^-, p_1^+, p_3^-, p_3^+ \right\|^{\mathrm{T}}, \qquad \mathbf{Q} = \left\| p_2^-, p_2^+ \right\|^{\mathrm{T}},
\end{aligned} \tag{39}$$

are the periodical functions from the circumferential coordinate φ, which can be expanded in the Fourier series in this coordinate

$$\begin{aligned}
[\mathbf{X}, \mathbf{P}] &= \sum_{n=0}^{\infty} \{ [\mathbf{X}_n(s), \mathbf{P}_n(s)] \cos n\varphi + [\mathbf{X}_{-n}(s), \mathbf{P}_{-n}(s)] \sin n\varphi \}, \\
[\mathbf{Z}, \mathbf{Q}] &= \sum_{n=0}^{\infty} \{ [\mathbf{Z}_n(s), \mathbf{Q}_n(s)] \sin n\varphi + [\mathbf{Z}_{-n}(s), \mathbf{Q}_{-n}(s)] \cos n\varphi \},
\end{aligned} \tag{40}$$

where $E_{ij}^\pm$, $E_{i3}^\pm$ and E_{33} are the tangential, transverse shear and transverse normal strains, correspondingly; $S_{i3}^\pm$ and P_{i3} are the generalized stress resultants having in this section a simplified form

[1] The joint influence of anisotropy and geometrical non-linearity on the stress–strain state of composite shells was analyzed by Grigolyuk and Kulikov (1981), and by Patel and Kennedy (1982).

$$
\begin{aligned}
&E_{11}^{\pm} = \frac{\partial v_1^{\pm}}{\partial s} + k_1 v_3^{\pm}, \qquad E_{22}^{\pm} = \frac{1}{r}\frac{\partial v_2^{\pm}}{\partial \varphi} + \rho v_1^{\pm} + k_2 v_3^{\pm}, \qquad E_{33} = \frac{1}{h}\left(v_3^{+} - v_3^{-}\right), \\
&E_{12}^{\pm} = \frac{\partial v_2^{\pm}}{\partial s} + \frac{1}{r}\frac{\partial v_1^{\pm}}{\partial \varphi} - \rho v_2^{\pm}, \qquad E_{i3}^{\pm} = \beta_i - \theta_i^{\pm}, \qquad \beta_i = \frac{1}{h}\left(v_i^{+} - v_i^{-}\right), \\
&\theta_1^{\pm} = k_1 v_1^{\pm} - \frac{\partial v_3^{\pm}}{\partial s}, \qquad \theta_2^{\pm} = k_2 v_2^{\pm} - \frac{1}{r}\frac{\partial v_3^{\pm}}{\partial \varphi}, \qquad \rho = \frac{1}{r}\frac{dr}{ds}, \\
&S_{i3}^{-} = H_{i3}^{-} - \widetilde{H}_{1i}^{00}\theta_1^{-} - \widetilde{H}_{1i}^{01}\theta_1^{+} - \widetilde{H}_{i2}^{00}\theta_2^{-} - \widetilde{H}_{i2}^{01}\theta_2^{+}, \\
&S_{i3}^{+} = H_{i3}^{+} - \widetilde{H}_{1i}^{10}\theta_1^{-} - \widetilde{H}_{1i}^{11}\theta_1^{+} - \widetilde{H}_{i2}^{10}\theta_2^{-} - \widetilde{H}_{i2}^{11}\theta_2^{+}, \qquad P_{i3} = T_{i3} + \widetilde{T}_{33}\beta_i, \\
&\widetilde{T}_{33} = \sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k} \widetilde{\sigma}_{33}^{(k)}\,\mathrm{d}z, \quad \widetilde{H}_{ij}^{\ell m} = \sum_{k=1}^{N}\int_{\delta_{k-1}}^{\delta_k} \widetilde{\sigma}_{ij}^{(k)}\left(N^{-}(z)\right)^{2-\ell-m}\left(N^{+}(z)\right)^{\ell+m}\mathrm{d}z, \qquad (\ell, m = 0, 1),
\end{aligned}
\tag{41}
$$

where r is the normal distance from the rotation axis to the reference surface.

Substituting the components of the state vectors and loading vectors from Eqs. (39) and (40) into the equilibrium equations (23), constitutive equations and formulas (41), and separating variables on sines and cosines, one can obtain two systems of linear differential equations

$$
\frac{\mathrm{d}\mathbf{Y}_n}{\mathrm{d}s} = \mathbf{F}_n(s, \mathbf{Y}_n, \mathbf{Y}_{-n}), \quad (n = 0, 1, \ldots), \tag{42}
$$

$$
\frac{\mathrm{d}\mathbf{Y}_{-n}}{\mathrm{d}s} = \mathbf{F}_{-n}(s, \mathbf{Y}_n, \mathbf{Y}_{-n}), \quad (n = 1, 2, \ldots), \tag{43}
$$

where $\mathbf{Y}_n$ and $\mathbf{Y}_{-n}$ are the vectors corresponding to the nth Fourier harmonic defined by

$$
\mathbf{Y}_n = \left\| H_{11,n}^{-}, H_{11,n}^{+}, H_{12,n}^{-}, H_{12,n}^{+}, S_{13,n}^{-}, S_{13,n}^{+}, v_{1,n}^{-}, v_{1,n}^{+}, v_{2,n}^{-}, v_{2,n}^{+}, v_{3,n}^{-}, v_{3,n}^{+} \right\|^{\mathrm{T}}, \quad (n = 0, \pm 1, \ldots). \tag{44}
$$

The systems of the differential Eqs. (42) and (43) should be solved together, because for anisotropic shells of revolution the two sets of symmetric and antisymmetric displacements, strains and stress resultants, associated with each Fourier harmonic (40), are coupled. So, the order of the governing system is redoubled and equals 24. This strongly complicates the solution of the non-axisymmetric problem for multilayered anisotropic shells. There are not such difficulties for multilayered orthotropic shells, where the similar systems of the differential Eqs. (42) and (43) are not coupled and can be solved by more simple methods.

The boundary conditions of the non-axisymmetric problem can be written as follows:

$$
\begin{aligned}
&Y_{n,m}(s^{-})\ell_m + Y_{n,m+6}(s^{-})(1 - \ell_m) = 0, \\
&Y_{n,m}(s^{+})\ell_{m+6} + Y_{n,m+6}(s^{+})(1 - \ell_{m+6}) = 0, \qquad (m = \overline{1,6}; n = 0, \pm 1, \ldots),
\end{aligned}
\tag{45}
$$

where $Y_{n,m}$ and $Y_{n,m+6}$ are the components of the state vectors $\mathbf{Y}_n$; ℓ_m and ℓ_{m+6} are the boundary coefficients, which may take the values 0 and 1.

So, the two-dimensional boundary value problem has been reduced to a sequence of the single-dimensional linear boundary value problems (42)–(45), which can be solved by using the above-mentioned discrete orthogonalization method.

As a numerical example, we consider the second tire problem concerning the non-axisymmetric response of an anisotropic tire subjected to inflation pressure and normal localized loading (Fig. 7). Let a four-layered anisotropic toroidal shell with the above geometrical and mechanical characteristics be subjected to uniform inflation pressure $q = 0.15$ MPa. This geometrically non-linear problem can be solved on the basis of the previous computational model. Then, the initially stressed shell is subjected to the normal localized loading (simulating the contact loading) distributed as follows: $p_3^{+} = -p_0\left[1 - (\varphi/\Delta)^4\right]$ if $-3.5 \leqslant s \leqslant 3.5$ mm

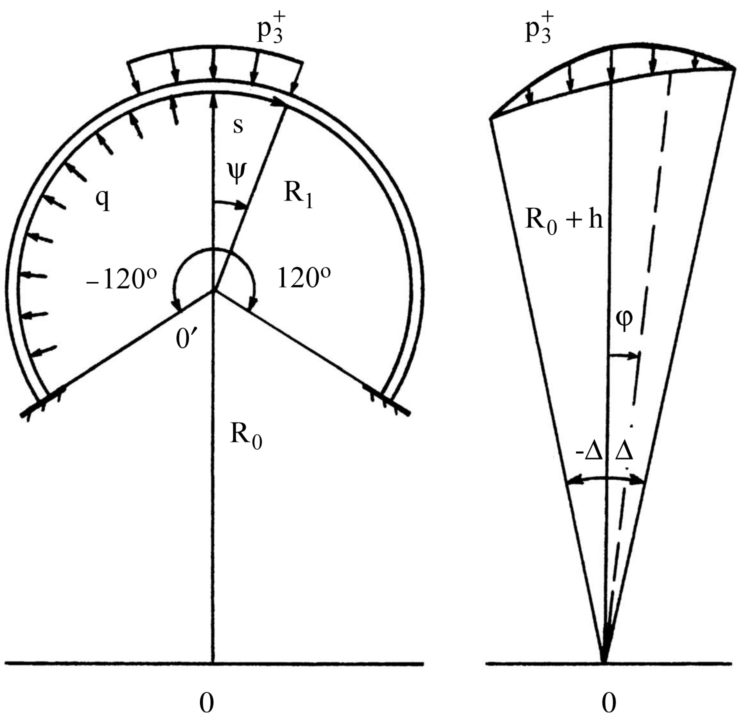

Fig. 7. Four-layered composite toroidal shell subjected to inflation pressure and normal localized loading.

and $-\Delta \leqslant \varphi \leqslant \Delta$, where $p_0 = 0.3125$ MPa and $\Delta = 0.4$. As already mentioned, in this approach the displacements are referred from a new reference surface which is computated by using the following assumptions: the effects of the meridian stretching and thickness variation are not included. Besides, the fast Fourier transform was used for the computation of the Fourier harmonics $p^+_{3,n}$ and $p^+_{3,-n}$ from Eqs. (39) and (40).

Figs. 8 and 9 show the distribution of the transverse stresses in the thickness direction at the various cross-sections of a tire. The solution presented here was obtained with at least 30 terms in each Fourier series and compared to results obtained using less terms. In most cases, there was a negligible difference between the 25-term solution and 30-term solution. As expected, the influence of anisotropy is important both inside and outside the contact zone. We can see that the symmetry conditions are not satisfied here especially at $\varphi = \pm 22°$ and $\pm 24°$, i. e., nearer to the contact region.

It should be noted that due to the essentially non-uniform distribution of the transverse shear stresses σ_{13} and σ_{23} over the thickness of a tire, the known first-order global approximation theories (Grigolyuk and Kulikov, 1988b; Kulikov, 1996; Noor and Burton, 1990) do not provide the reliable prediction of tire failure, since in these theories the transverse shear stresses are distributed in the thickness direction according to the simplest parabolic law.

6. Conclusion

The refined first-order global approximation theory of prestressed multilayered anisotropic shells has been developed. The effects of the laminated anisotropic material response, initially stressed state response,

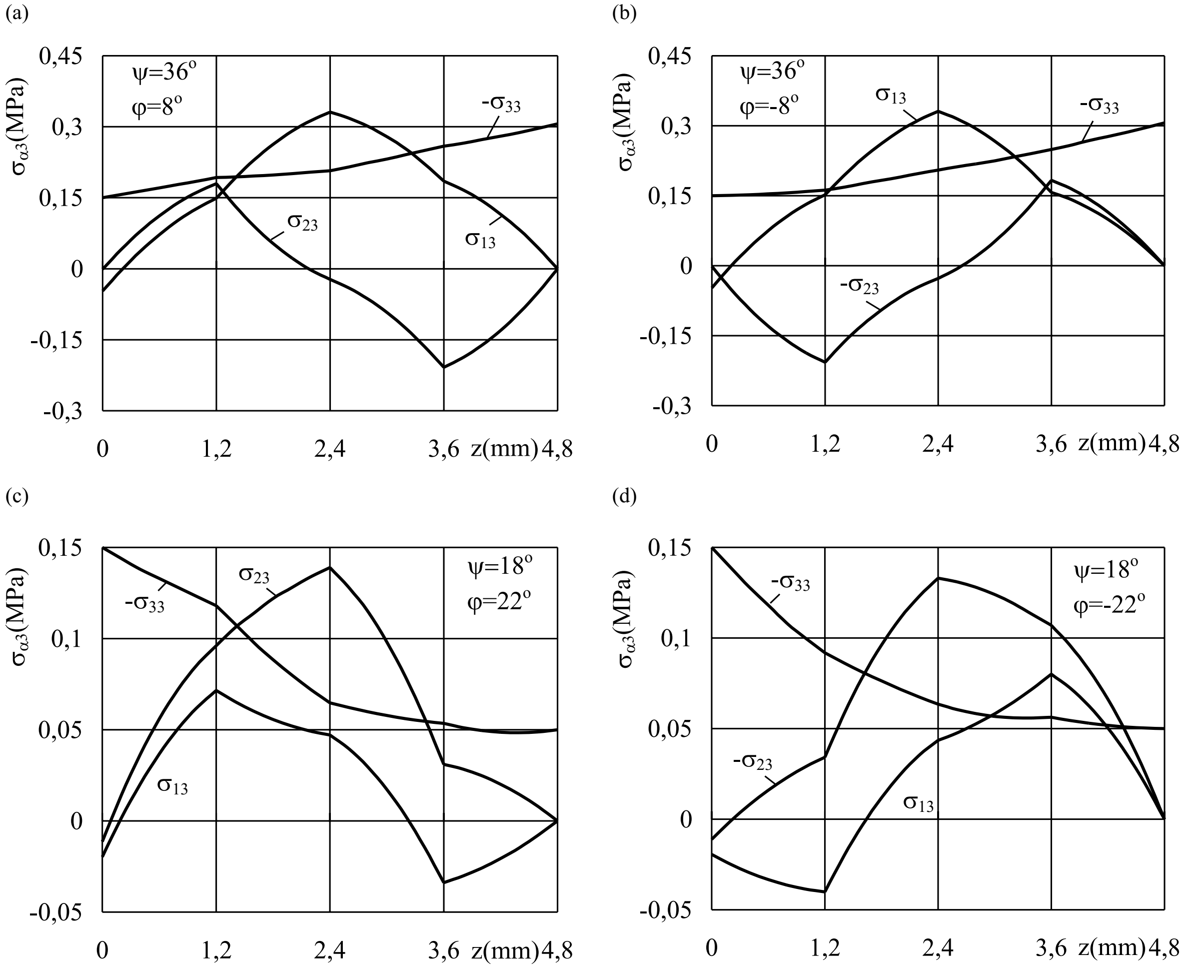

Fig. 8. Distribution of transverse shear and normal stresses in thickness direction inside contact region at: (a) $\psi = 36°$ and $\varphi = 8°$; (b) $\psi = 36°$ and $\varphi = -8°$; (c) $\psi = 18°$ and $\varphi = 22°$ and (d) $\psi = 18°$ and $\varphi = -22°$.

geometrical non-linearity, transverse shear and transverse normal strains are included. This theory is based on the refined kinematic Timoshenko hypothesis adopted for the displacement vector. As unknown functions, the tangential and transverse displacements of the face surfaces of the shell have been chosen. Such choice of unknowns allows as much as possible to algorithmize the computational modeling of a series of important shell problems. The developed theory can be used for solving the shell problems where the above effects are significant. Such problems can be met in many fields of the engineering science and especially in the tire mechanics.

The governing equations of the theory of prestressed multilayered anisotropic shells have been obtained by using the principle of the virtual work and well-known partially non-linear Novozhilov's strain–displacement relationships. It is important that the equilibrium equations of the three-dimensional non-linear elasticity theory are satisfied pointwise into the shell body.

Two computational models for solving the axisymmetric and non-axisymmetric problems of prestressed multilayered anisotropic shells of revolution have been presented. The first computational model is based

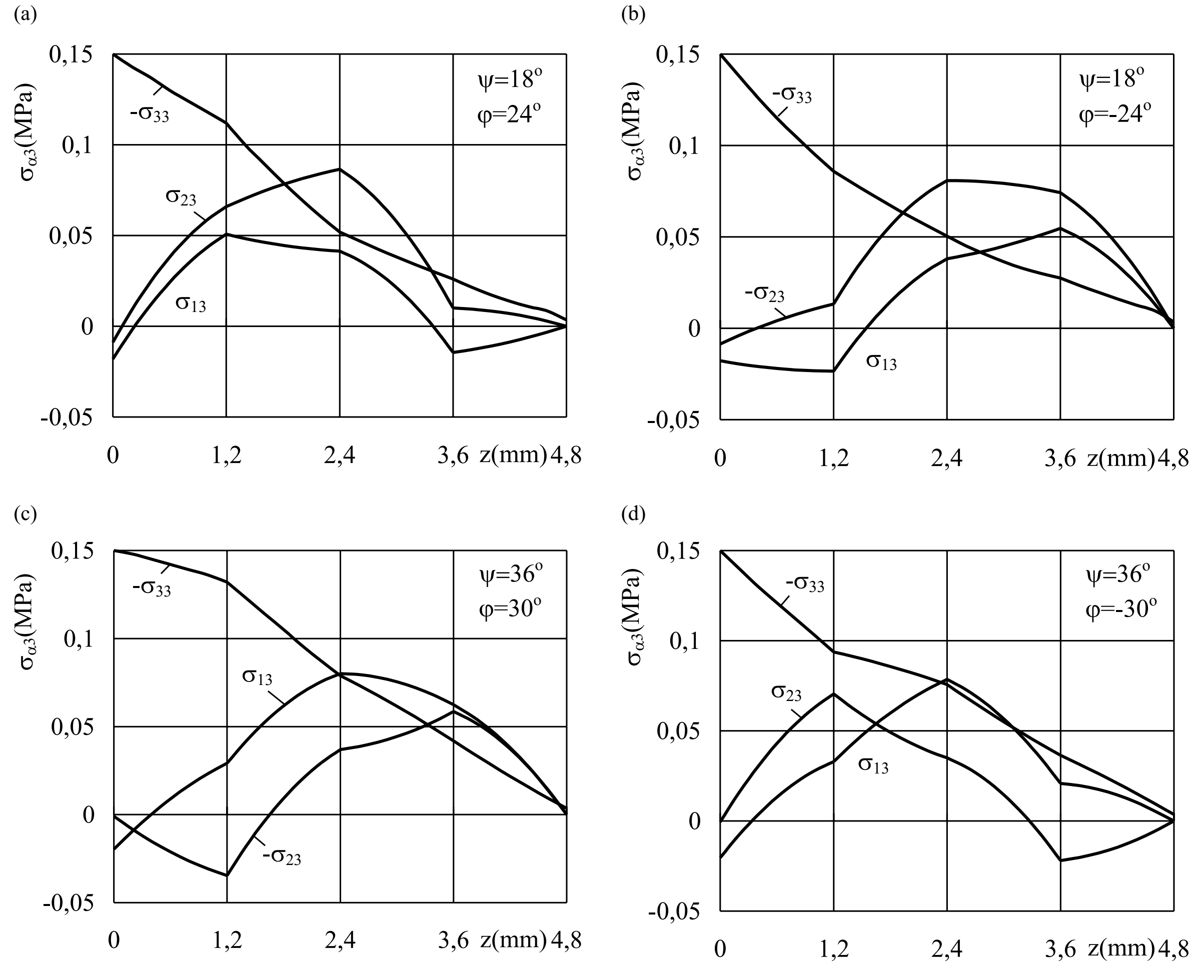

Fig. 9. Distribution of transverse shear and normal stresses in thickness direction outside contact region at: (a) $\psi = 18°$ and $\varphi = 24°$; (b) $\psi = 18°$ and $\varphi = -24°$; (c) $\psi = 36°$ and $\varphi = 30°$ and (d) $\psi = 36°$ and $\varphi = -30°$.

on the Newton–Raphson method and the incremental method by using the discrete orthogonalization method. For example, a relatively simple problem of the non-linear axisymmetric response of the anisotropic bias-ply tire has been solved. The tire is modeled by the four-layered cross-ply toroidal shell subjected to inflation pressure. This non-linear problem has been solved by using the Newton–Raphson method and the incremental method. It has been shown that both numerical solutions give similar results, excepting the values of the transverse shear stresses at the inner surface of a tire. It has also been established that neglecting the effects of anisotropy, and geometrical non-linearity can lead to an incorrect description of the stress field in cross-ply toroidal shells.

The second computational model is based on the expansion of the unknown functions and external loads in the Fourier series in the circumferential coordinate, and using the fast Fourier transform. After separation of variables on sines and cosines, two systems of ordinary differential equations have been derived. These systems have also been solved by using the discrete orthogonalization method. As an example, the non-axisymmetric response of the prestressed anisotropic bias-ply tire has been studied. The tire is modeled by the four-layered cross-ply toroidal shell subjected to inflation pressure and localized loading, simulating contact pressure and acting on the outer surface.

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