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Non-linear analysis of multilayered shells under initial stress

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Abstract

A refined non-linear first-order discrete-layer theory of initially stressed composite shells is developed. The material of each layer of the shell is assumed to be linearly elastic, anisotropic, homogeneous or fiber-reinforced. The transverse shear and transverse normal effects are included. It is also assumed, that the well-known Novozhilov's three-dimensional partially non-linear strain–displacement relationships are valid. As unknown functions the tangential and transverse displacements of external surfaces of the shell and layer interfaces are selected. A computational model for solving the non-linear problems of the axisymmetric deformation of initially stressed multilayered anisotropic shells of revolution is presented. The joint influence of anisotropy, initially stressed state response, geometrical non-linearity and laminated material response on the stress state of the shell is examined. Results show, that neglecting the effects of anisotropy and geometrical non-linearity leads to an incorrect description of the stress field in cross-ply toroidal shells made of cord-rubber composites. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Elasticity; Anisotropy; Multilayered shell; Initial stresses

1. Introduction

In recent years, considerable interest has been found in the concerned literature to substantiate the geometrically non-linear theory of elastic multilayered composite shells and plates. In this context, a number of monographs and survey papers [1–6] where rich references of the literature dealing with similar problems to the ones in our study can be found are indicated. For some works addressing the problem of multilayered composite plates and shells under initial stress, the reader is referred to Sun [7], Biot [8], Sun and Whitney [9], Kulikov [10], Grigolyuk and Kulikov [11] and Kulikov [12].

Here, the refined discrete-layer theory of initially stressed anisotropic shells is developed. Discrete-layer theories are based on layer-by-layer approximations of the displacements, strains or stresses [4,5]. Consequently, the order of the governing equations is dependent on the number of layers of the shell. The simplest examples of these theories are the so-called first-order discrete-layer theories [12–17][1] based on Grigolyuk's zig-zag hypothesis (piecewise linear approximation) for displacements in the thickness direction.

The direct use of the traditional first-order discrete-layer theory [3,12,17] for solving a series of important shell problems such as the contact

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[1] Note that these theories are extensions of Grigolyuk's theory for three-layered shells [18] and multilayered shallow shells [19,20], where the well-known zig-zag hypothesis was first formulated.

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problems is not always convenient. In these problems, it is more convenient to select as unknown functions the tangential and transverse displacements of the face surfaces of the shell, since with the help of these displacements the kinematic requirement of no penetration of the contact bodies can be fulfilled.

This theory is based on the refined Grigolyuk's hypothesis for the displacement vector. The governing equations of the theory of initially stressed multilayered anisotropic shells are obtained by using the principle of the virtual work and Novozhilov's partially non-linear strain–displacement relationships. An outcome of this approach is that the equilibrium equations of the geometrically nonlinear elasticity theory are satisfied pointwise into the shell body with an exactitude acceptable for the thin shell structures.

On the basis of the proposed discrete-layer theory the computational model for solving the axisymmetric problems of initially stressed multilayered anisotropic shells of revolution is elaborated. The material of each layer of the shell is assumed to be linearly elastic, anisotropic, homogeneous or fiber-reinforced, such that in each point there is a single surface of elastic symmetry parallel to the reference surface. The adopted computational model is based on the Newton–Raphson method and the incremental method. The linear boundary value problem is solved by applying the discrete orthogonalization method [3].

An example includes some relatively simple problem, namely, the non-linear axisymmetric response of a cross-ply toroidal shell made of cord-rubber materials and subjected to inflation pressure. Numerical results show that the joint influence of anisotropy and geometrical nonlinearity on the stress field in composite toroidal shells is essential.

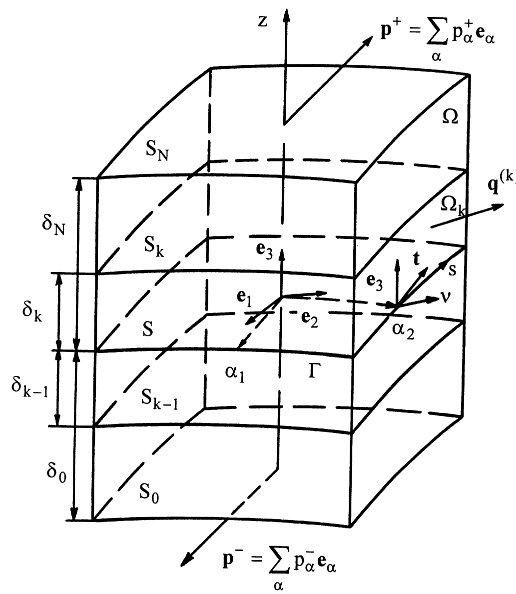

Fig. 1. The element of the multilayered shell.

2. Elasticity theory of initially stressed multilayered shells

Let us consider the shell built up in the general case by the arbitrary superposition across the wall thickness of N thin layers of uniform thickness h_k. The kth layer may be defined as a three-dimensional body of volume V_k bounded by two surfaces S_{k-1} and S_k, located at the distances δ_{k-1} and δ_k measured with respect to the reference surface S, and the edge boundary surface Ω_k that is perpendicular to the reference surface (see Fig. 1). The full-edge boundary surface $\Omega = \sum_{k=1}^{N} \Omega_k$ is generated by the normals to the reference surface along the bounding curve Γ (with the arc length s) of this surface. It is also assumed that the bounding surfaces S_{k-1} and S_k are continuous, sufficiently smooth and without any singularities. Let the reference surface be referred to an orthogonal curvilinear coordinate system α_1 and α_2 which coincides with the lines of principal curvatures of its surface. The z-axis is oriented along the outward unit vector $\mathbf{e}_3$ normal to the reference surface.

The constituent layers of the shell are supposed to be rigidly joined, so that no slip on contact surfaces and no separation of layers can occur. The material of each constituent layer is assumed to be linearly elastic, anisotropic, homogeneous or fiber-reinforced, such that in each point there is a single surface of elastic symmetry parallel to the reference surface (monoclinic symmetry). Let $\tilde{\sigma}_{\alpha\beta}^{(k)}$ are the initial stresses; $\tilde{p}_{\alpha}^{-}$ and $\tilde{p}_{\alpha}^{+}$ are the intensities of the initial external loading acting on the bottom surface S_0 and top surface S_N in the α_1, α_2 and z coordinate directions, respectively; $\tilde{q}_v^{(k)}$, $\tilde{q}_t^{(k)}$ and $\tilde{q}_3^{(k)}$ are the intensities of the initial external loading acting on the edge boundary surface Ω_k in the v, t and z directions, where $\mathbf{v}$ and $\mathbf{t}$ are the normal and tangential unit vectors to the bounding curve Γ (see Fig. 1). Here and in the following developments the index k identifies the belonging of any quantity to the kth layer ($k = \overline{1, N}$) and indices α, β take the values 1, 2, 3, and indices i, j take the values 1, 2.

The boundary value problem for the prestressed multilayered shell is defined by setting the additional loading $p_\alpha^-, p_\alpha^+, q_v^{(k)}, q_t^{(k)}, q_3^{(k)}$ (see Fig. 1). As a result of this loading the resulting stress state can be represented as

$$\sigma_{\alpha\beta}^{(k)r} = \tilde{\sigma}_{\alpha\beta}^{(k)} + \sigma_{\alpha\beta}^{(k)}, \tag{1}$$

where $\sigma_{\alpha\beta}^{(k)}$ are the additional stresses of the kth layer.

Since the initial surface loads $\tilde{p}_\alpha^-$, $\tilde{p}_\alpha^+$, $\tilde{q}_v^{(k)}$, $\tilde{q}_t^{(k)}$, $\tilde{q}_3^{(k)}$ and initial stresses $\tilde{\sigma}_{\alpha\beta}^{(k)}$ constitute the self-equilibrated system and assuming the case of thin shells, the principle of the virtual work for the prestressed thin multilayered shell can be written in the following form [21]:

$$\begin{aligned}\sum_{k=1}^{N} \iiint_{V_k} \sum_{\alpha \leqslant \beta} (\sigma_{\alpha\beta}^{(k)} \delta e_{\alpha\beta}^{(k)} + \sigma_{\alpha\beta}^{(k)r} \delta \eta_{\alpha\beta}^{(k)}) A_1 A_2 \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}z \\ - \iint_{S_N} \sum_{\alpha} p_\alpha^+ \delta u_\alpha^{(N)} \, \mathrm{d}S + \iint_{S_0} \sum_{\alpha} p_\alpha^- \delta u_\alpha^{(1)} \, \mathrm{d}S \\ + \sum_{n=1}^{N-1} \iint_{S_n} \sum_{\alpha} \tau_\alpha^{(n)} (\delta u_\alpha^{(n+1)} - \delta u_\alpha^{(n)}) \, \mathrm{d}S \\ - \sum_{k=1}^{N} \iint_{\Omega_k} (q_v^{(k)} \delta u_v^{(k)} + q_t^{(k)} \delta u_t^{(k)} + q_3^{(k)} \delta u_3^{(k)}) \, \mathrm{d}S = 0,\end{aligned} \tag{2}$$

where A_1 and A_2 are the Lamé coefficients of the reference surface; $u_\alpha^{(k)}$ are the components of the displacement vector of the kth layer in the coordinate system α_1, α_2, z that are refferred from the reference surface $S; u_v^{(k)}, u_t^{(k)}$ and $u_3^{(k)}$ are the components of the displacement vector of the kth layer in the coordinate system $v, t, z; \tau_\alpha^{(n)}$ are the interlaminar transverse stresses acting on the layer interfaces S_n, and $e_{\alpha\beta}^{(k)}$ and $\eta_{\alpha\beta}^{(k)}$ are the linear and non-linear parts of strains of the kth layer, i.e.

$$\varepsilon_{\alpha\beta}^{(k)} = e_{\alpha\beta}^{(k)} + \eta_{\alpha\beta}^{(k)}.$$

The three-dimensional Novozhilov's partially non-linear strain–displacement relationships in the Lagrange description for the multilayered thin shells will be [3]

$$\begin{aligned} e_{11}^{(k)} &= \frac{1}{A_1} \frac{\partial u_1^{(k)}}{\partial \alpha_1} + B_2 u_2^{(k)} + k_1 u_3^{(k)}, \\ e_{12}^{(k)} &= \frac{1}{A_1} \frac{\partial u_2^{(k)}}{\partial \alpha_1} + \frac{1}{A_2} \frac{\partial u_1^{(k)}}{\partial \alpha_2} - B_2 u_1^{(k)} - B_1 u_2^{(k)}, \\ e_{13}^{(k)} &= \frac{\partial u_1^{(k)}}{\partial z} + \Theta_1^{(k)}, \quad e_{33}^{(k)} = \frac{\partial u_3^{(k)}}{\partial z}, \\ \Theta_1^{(k)} &= \frac{1}{A_1} \frac{\partial u_3^{(k)}}{\partial \alpha_1} - k_1 u_1^{(k)} \quad (1 \Leftrightarrow 2), \\ \eta_{11}^{(k)} &= \frac{1}{2} (\Theta_1^{(k)})^2, \quad \eta_{12}^{(k)} = \Theta_1^{(k)} \Theta_2^{(k)}, \quad \eta_{13}^{(k)} = 0, \\ \eta_{33}^{(k)} &= \frac{1}{2} \left(\frac{\partial u_1^{(k)}}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial u_2^{(k)}}{\partial z} \right)^2, \\ B_1 &= \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \quad (1 \Leftrightarrow 2), \end{aligned} \tag{3}$$

where k_1 and k_2 are the principal curvatures of the reference surface. In the expressions of tangential strains (3) only those non-linear geometrical terms that depend on $\Theta_1^{(k)}$ and $\Theta_2^{(k)}$ are retained. The remaining non-linear terms are discarded. Sign ($1 \Leftrightarrow 2$) accompanying certain relations means that the remaining relations, not explicitly written, are correspondingly obtained by replacing subscript 1 by 2 and vice versa.

The governing equations of the geometrically non-linear elasticity theory for the prestressed thin multilayered shell can be derived by applying the principle of the virtual work (2). Substituting the strain–displacement relationships (3) into Eq. (2) and using Gauss' theorem, one obtains the following variational equation:

$$\sum_{k=1}^{N}\iiint_{V_k}\sum_{\alpha} L_\alpha^{(k)}\delta u_\alpha^{(k)} A_1 A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}z - \iint_{S_N}\sum_{\alpha}(s_{\alpha 3}^{(N)} - p_\alpha^+)\delta u_\alpha^{(N)}\,\mathrm{d}S + \iint_{S_0}\sum_{\alpha}(s_{\alpha 3}^{(1)} - p_\alpha^-)\delta u_\alpha^{(1)}\,\mathrm{d}S + \sum_{n=1}^{N-1}\iint_{S_n}\sum_{\alpha}[(s_{\alpha 3}^{(n+1)} - \tau_\alpha^{(n)})\delta u_\alpha^{(n+1)} - (s_{\alpha 3}^{(n)} - \tau_\alpha^{(n)})\delta u_\alpha^{(n)}]\,\mathrm{d}S - \sum_{k=1}^{N}\iint_{\Omega_k}[(\sigma_{vv}^{(k)} - q_v^{(k)})\delta u_v^{(k)} + (\sigma_{vt}^{(k)} - q_t^{(k)})\delta u_t^{(k)} + (\Sigma_{v3}^{(k)} - q_3^{(k)})\delta u_3^{(k)}]\,\mathrm{d}S = 0, \tag{4}$$

where $\sigma_{vv}^{(k)}$, $\sigma_{vt}^{(k)}$, $\sigma_{v3}^{(k)}$ are the stress tensor components of the kth layer in the coordinate system v, t, z; $\Sigma_{i3}^{(k)}$ and $s_{i3}^{(k)}$ are the generalized transverse shear stresses, and $L_\alpha^{(k)}$ are the three-dimensional non-linear differential operators, corresponding to Novozhilov's strain–displacement relationships (3), which can be written as follows:

$$L_1^{(k)} = \frac{1}{A_1}\frac{\partial \sigma_{11}^{(k)}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial \sigma_{12}^{(k)}}{\partial \alpha_2} + \frac{\partial s_{13}^{(k)}}{\partial z} + B_1(\sigma_{11}^{(k)} - \sigma_{22}^{(k)}) + 2B_2\sigma_{12}^{(k)} + k_1\Sigma_{13}^{(k)} \quad (1\Leftrightarrow 2),$$

$$L_3^{(k)} = \frac{1}{A_1}\frac{\partial \Sigma_{13}^{(k)}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial \Sigma_{23}^{(k)}}{\partial \alpha_2} + \frac{\partial s_{33}^{(k)}}{\partial z} + B_1\Sigma_{13}^{(k)} + B_2\Sigma_{23}^{(k)} - k_1\sigma_{11}^{(k)} - k_2\sigma_{22}^{(k)},$$

$$\Sigma_{i3}^{(k)} = \sigma_{i3}^{(k)} + \Theta_1^{(k)}(\tilde{\sigma}_{1i}^{(k)} + \sigma_{1i}^{(k)}) + \Theta_2^{(k)}(\tilde{\sigma}_{i2}^{(k)} + \sigma_{i2}^{(k)}),$$

$$s_{i3}^{(k)} = \sigma_{i3}^{(k)} + \frac{\partial u_i^{(k)}}{\partial z}(\tilde{\sigma}_{33}^{(k)} + \sigma_{33}^{(k)}), \quad s_{33}^{(k)} = \sigma_{33}^{(k)}. \tag{5}$$

Equating to zero the coefficients of $\delta u_\alpha^{(k)}$, one obtains the equilibrium equations and boundary conditions of the geometrically non-linear elasticity theory of prestressed multilayered thin shells. These are

- the equilibrium equations for the kth layer:

$$L_\alpha^{(k)} = 0, \tag{6}$$

- the boundary conditions for the generalized transverse stresses on the top surface S_N:

$$s_{\alpha 3}^{(N)} = p_\alpha^+, \tag{7}$$

- the boundary conditions for the generalized transverse stresses on the bottom surface S_0:

$$s_{\alpha 3}^{(1)} = p_\alpha^-, \tag{8}$$

- the equilibrium conditions for the generalized transverse stresses at the layer interfaces S_n:

$$s_{\alpha 3}^{(n+1)} = s_{\alpha 3}^{(n)} = \tau_\alpha^{(n)} \quad (n = \overline{1, N-1}), \tag{9}$$

- the boundary conditions on the edge boundary surfaces Ω_k:

$$\sigma_{vv}^{(k)} = q_v^{(k)}, \quad \sigma_{vt}^{(k)} = q_t^{(k)}, \quad \Sigma_{v3}^{(k)} = q_3^{(k)}. \tag{10}$$

Additionally, we should invoke the generalized Hooke's law:

$$\begin{aligned}
\sigma_{11}^{(k)} &= b_{11}^{(k)}\varepsilon_{11}^{(k)} + b_{12}^{(k)}\varepsilon_{22}^{(k)} + \underline{b_{13}^{(k)}\varepsilon_{33}^{(k)}} + b_{16}^{(k)}\varepsilon_{12}^{(k)},\\
\sigma_{22}^{(k)} &= b_{12}^{(k)}\varepsilon_{11}^{(k)} + b_{22}^{(k)}\varepsilon_{22}^{(k)} + \underline{b_{23}^{(k)}\varepsilon_{33}^{(k)}} + b_{26}^{(k)}\varepsilon_{12}^{(k)},\\
\sigma_{33}^{(k)} &= b_{13}^{(k)}\varepsilon_{11}^{(k)} + b_{23}^{(k)}\varepsilon_{22}^{(k)} + b_{33}^{(k)}\varepsilon_{33}^{(k)} + b_{36}^{(k)}\varepsilon_{12}^{(k)},\\
\sigma_{12}^{(k)} &= b_{16}^{(k)}\varepsilon_{11}^{(k)} + b_{26}^{(k)}\varepsilon_{22}^{(k)} + \underline{b_{36}^{(k)}\varepsilon_{33}^{(k)}} + b_{66}^{(k)}\varepsilon_{12}^{(k)},\\
\sigma_{23}^{(k)} &= b_{44}^{(k)}\varepsilon_{23}^{(k)} + b_{45}^{(k)}\varepsilon_{13}^{(k)}, \quad \sigma_{13}^{(k)} = b_{45}^{(k)}\varepsilon_{23}^{(k)} + b_{55}^{(k)}\varepsilon_{13}^{(k)},
\end{aligned} \tag{11}$$

where $b_{\ell m}^{(k)}$ are the stiffness coefficients of the kth layer ($\ell, m = \overline{1,6}$).

So, we have all fundamental relationships (1), (3), (6)–(11) for finding the resulting stress state of the prestressed multilayered anisotropic shell.

3. Discrete-layer theory of initially stressed shells

The first-order discrete-layer theory of shells is based on the piecewise linear approximation for the displacement vector in the thickness direction

$$u_\alpha^{(k)} = N_k^-(z)v_\alpha^{(k-1)} + N_k^+(z)v_\alpha^{(k)},$$

$$N_k^-(z) = (\delta_k - z)/h_k, \quad N_k^+(z) = (z - \delta_{k-1})/h_k, \tag{12}$$

where $v_\alpha^{(k-1)}(\alpha_1, \alpha_2)$ and $v_\alpha^{(k)}(\alpha_1, \alpha_2)$ are the tangential and transverse displacements of the face surfaces of the shell and layer interfaces, $N_k^-(z)$ and

$N_k^+(z)$ are the linear shape functions of the kth layer. The piecewise linear approximation (12) may be considered as a refined Grigolyuk's hypothesis (see, for example, the monograph [3], where as unknown functions the displacements of the reference surface and rotation components for the kth layer are selected). The advantage of the proposed approach is obvious, since with the help of the displacements $v_\alpha^{(0)}$ and $v_\alpha^{(N)}$ the kinematic boundary conditions on the face surfaces of the shell, and in particular, the conditions of no penetration of the contact bodies can be formulated. Besides, this provides a convenient way to express the non-linear strain–displacement relationships in terms of layer interfaces strains.

Substituting the displacements from Eq. (12) into the strain–displacement relationships (3) and variational equation (4), and taking into account that a shell is thin, the following equations of the geometrically non-linear discrete-layer theory of prestressed shells are obtained:

- the equilibrium equations

$$\int_{\delta_0}^{\delta_1} L_\alpha^{(1)} N_1^-(z)\,dz = 0, \quad \int_{\delta_{N-1}}^{\delta_N} L_\alpha^{(N)} N_1^+(z)\,dz = 0,$$

$$\int_{\delta_{n-1}}^{\delta_n} L_\alpha^{(n)} N_n^+(z)\,dz + \int_{\delta_n}^{\delta_{n+1}} L_\alpha^{(n+1)} N_{n+1}^-(z)\,dz = 0$$

$$(n = \overline{1, N-1}), \tag{13}$$

- the boundary conditions for the generalized transverse stresses on the top surface (7),
- the boundary conditions for the generalized transverse stresses on the bottom surface (8),
- the equilibrium conditions for the generalized transverse stresses at the layer interfaces (9),
- the natural boundary conditions on the edge boundary surface Ω:

$$(H_{vr}^{(1)-} - \hat{H}_{vr}^{(1)-})\,\delta v_r^{(0)} = 0,$$

$$(H_{vr}^{(N)+} - \hat{H}_{vr}^{(N)+})\,\delta v_r^{(N)} = 0,$$

$$(H_{vr}^{(n)+} + H_{vr}^{(n+1)-} - \hat{H}_{vr}^{(n)+} - \hat{H}_{vr}^{(n+1)-})\,\delta v_r^{(n)} = 0$$

$$(r = v, t;\ n = \overline{1, N-1}),$$

$$(S_{v3}^{(1)-} - \hat{H}_{v3}^{(1)-})\,\delta v_3^{(0)} = 0,$$

$$(S_{v3}^{(N)+} - \hat{H}_{v3}^{(N)+})\,\delta v_3^{(N)} = 0,$$

$$(S_{v3}^{(n)+} + S_{v3}^{(n+1)-} - \hat{H}_{v3}^{(n)+} - \hat{H}_{v3}^{(n+1)-})\,\delta v_3^{(n)} = 0$$

$$(n = \overline{1, N-1}), \tag{14}$$

where $H_{vv}^{(k)\pm}$, $H_{vt}^{(k)\pm}$, $S_{v3}^{(k)\pm}$ are the generalized stress resultants:

$$H_{vv}^{(k)\pm} = \int_{\delta_{k-1}}^{\delta_k} \sigma_{vv}^{(k)} N_k^\pm(z)\,dz,$$

$$H_{vt}^{(k)\pm} = \int_{\delta_{k-1}}^{\delta_k} \sigma_{vt}^{(k)} N_k^\pm(z)\,dz,$$

$$S_{v3}^{(k)\pm} = \int_{\delta_{k-1}}^{\delta_k} \Sigma_{v3}^{(k)} N_k^\pm(z)\,dz, \tag{15}$$

and $\hat{H}_{vv}^{(k)\pm}$, $\hat{H}_{vt}^{(k)\pm}$, $\hat{H}_{v3}^{(k)\pm}$ are the generalized loading resultants that are obtained from Eq. (15) by replacing the stresses $\sigma_{vv}^{(k)}$, $\sigma_{vt}^{(k)}$, $\Sigma_{v3}^{(k)}$ by intensities of the external loads $q_v^{(k)}$, $q_t^{(k)}$, $q_3^{(k)}$ acting in the v, t, z directions, correspondingly;

- the strain–displacement relationships

$$\varepsilon_{ij}^{(k)} = N_k^-(z) E_{ij}^{(k-1)} + N_k^+(z) E_{ij}^{(k)},$$

$$\varepsilon_{i3}^{(k)} = N_k^-(z) E_{i3}^{(k)-} + N_k^+(z) E_{i3}^{(k)+},$$

$$\varepsilon_{33}^{(k)} = E_{33}^{(k)} = \beta_3^{(k)} + \tfrac{1}{2}(\beta_1^{(k)})^2 + \tfrac{1}{2}(\beta_2^{(k)})^2, \tag{16}$$

where $E_{ij}^{(k-1)}$, $E_{i3}^{(k)-}$ and $E_{ij}^{(k)}$, $E_{i3}^{(k)+}$ are the tangential and transverse shear strains of the bottom and top surfaces of the kth layer, respectively:

$$E_{11}^{(\ell)} = e_{11}^{(\ell)} + \tfrac{1}{2}(\theta_1^{(\ell)})^2, \quad E_{12}^{(\ell)} = e_{12}^{(\ell)} + \theta_1^{(\ell)}\theta_2^{(\ell)},$$

$$e_{11}^{(\ell)} = \frac{1}{A_1}\frac{\partial v_1^{(\ell)}}{\partial \alpha_1} + B_2 v_2^{(\ell)} + k_1 v_3^{(\ell)},$$

$$e_{12}^{(\ell)} = \frac{1}{A_1}\frac{\partial v_2^{(\ell)}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial v_1^{(\ell)}}{\partial \alpha_2} - B_2 v_1^{(\ell)} - B_1 v_2^{(\ell)},$$

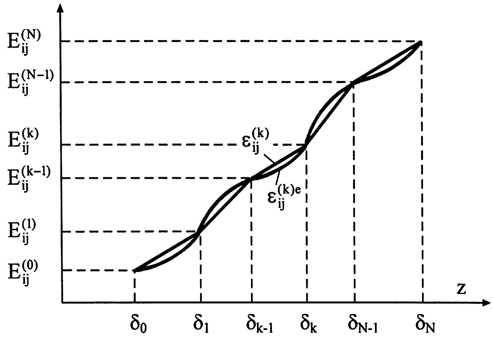

Fig. 2. The distribution of the tangential strains $\varepsilon_{ij}^{(k)}$ and $\varepsilon_{ij}^{(k)e}$ over the shell thickness.

$$\theta_1^{(\ell)} = k_1 v_1^{(\ell)} - \frac{1}{A_1}\frac{\partial v_3^{(\ell)}}{\partial \alpha_1} \quad (1\Leftrightarrow 2) \quad (\ell = \overline{0,N}),$$

$$E_{i3}^{(k)-} = \beta_i^{(k)} - \theta_i^{(k-1)}, \quad E_{i3}^{(k)+} = \beta_i^{(k)} - \theta_i^{(k)},$$

$$\beta_\alpha^{(k)} = \frac{1}{h_k}(v_\alpha^{(k)} - v_\alpha^{(k-1)}). \tag{17}$$

Note that the tangential strains $\varepsilon_{ij}^{(k)}$ are distributed over the shell thickness according to the piecewise linear law since the refined Grigolyuk's zig-zag hypothesis has been adopted. As can be seen from Fig. 2, it is an acceptable assumption for the thin shell structures. Really, better expressions for the tangential strains can be written by using the piecewise quadratic approximation that are exact for the proposed non-linear shell theory, i.e.

$$\varepsilon_{11}^{(k)e} = N_k^-(z)e_{11}^{(k-1)} + N_k^+(z)e_{11}^{(k)} + \tfrac{1}{2}(\Theta_1^{(k)})^2,$$

$$\varepsilon_{12}^{(k)e} = N_k^-(z)e_{12}^{(k-1)} + N_k^+(z)e_{12}^{(k)} + \Theta_1^{(k)}\Theta_2^{(k)},$$

$$\Theta_1^{(k)} = -N_k^-(z)\theta_1^{(k-1)} - N_k^+(z)\theta_1^{(k)} \quad (1\Leftrightarrow 2).$$

It is apparent that from the last equations and Eqs. (16), (17) follow that the coupling conditions $\varepsilon_{ij}^{(k)e}(\delta_m) = \varepsilon_{ij}^{(k)}(\delta_m) = E_{ij}^{(m)}$ are satisfied, where $m = k-1$ and k. Besides, values of these strains will always coincide for the geometrically linear shell theory.

Introducing the non-linear differential operators corresponding to the first-order discrete-layer theory

$$\Im_1^{(k)\pm} = \frac{1}{A_1}\frac{\partial H_{11}^{(k)\pm}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial H_{12}^{(k)\pm}}{\partial \alpha_2} + B_1(H_{11}^{(k)\pm} - H_{22}^{(k)\pm}) + 2B_2 H_{12}^{(k)\pm} + k_1 S_{13}^{(k)\pm} \mp \frac{1}{h_k}P_{13}^{(k)} \quad (1\Leftrightarrow 2),$$

$$\Im_3^{(k)\pm} = \frac{1}{A_1}\frac{\partial S_{13}^{(k)\pm}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial S_{23}^{(k)\pm}}{\partial \alpha_2} + B_1 S_{13}^{(k)\pm} + B_2 S_{23}^{(k)\pm} - k_1 H_{11}^{(k)\pm} - k_2 H_{22}^{(k)\pm} \mp \frac{1}{h_k}T_{33}^{(k)}, \tag{18}$$

where $H_{i\alpha}^{(k)\pm}$, $S_{i3}^{(k)\pm}$, $P_{i3}^{(k)}$ are the generalized stress resultants and $T_{\alpha 3}^{(k)}$ are the classical stress resultants defined as

$$H_{i\alpha}^{(k)\pm} = \int_{\delta_{k-1}}^{\delta_k} \sigma_{i\alpha}^{(k)} N_k^\pm(z)\,\mathrm{d}z,$$

$$S_{i3}^{(k)\pm} = \int_{\delta_{k-1}}^{\delta_k} \Sigma_{i3}^{(k)} N_k^\pm(z)\,\mathrm{d}z,$$

$$P_{i3}^{(k)} = \int_{\delta_{k-1}}^{\delta_k} s_{i3}^{(k)}\,\mathrm{d}z, \quad T_{\alpha 3}^{(k)} = \int_{\delta_{k-1}}^{\delta_k} \sigma_{\alpha 3}^{(k)}\,\mathrm{d}z,$$

$$\Sigma_{i3}^{(k)} = \sigma_{i3}^{(k)} + \Theta_1^{(k)}(\tilde{\sigma}_{1i}^{(k)} + \sigma_{1i}^{(k)}) + \Theta_2^{(k)}(\tilde{\sigma}_{i2}^{(k)} + \sigma_{i2}^{(k)}),$$

$$\Theta_i^{(k)} = -N_k^-(z)\theta_i^{(k-1)} - N_k^+(z)\theta_i^{(k)},$$

$$s_{i3}^{(k)} = \sigma_{i3}^{(k)} + \beta_i^{(k)}(\tilde{\sigma}_{33}^{(k)} + \sigma_{33}^{(k)}), \tag{19}$$

taking into account relations (7)–(9) and integrating by parts, one can obtain from Eqs. (13), $3(N+1)$ non-linear equilibrium equations of the initially stressed multilayered thin shell in terms of stress resultants

$$\Im_\alpha^{(1)-} - p_\alpha^- = 0, \quad \Im_\alpha^{(N)+} + p_\alpha^+ = 0,$$

$$\Im_\alpha^{(n)+} + \Im_\alpha^{(n+1)-} = 0 \quad (n = \overline{1, N-1}). \tag{20}$$

In order to obtain the constitutive equations for the stress resultants, the equations of the generalized Hooke's law (11) should be used. Unfortunately, such approach cannot correctly describe the shells made of incompressible materials or nearly incompressible materials having Poisson's coefficients

$\nu_{\alpha\beta} \approx 0.5$ ($\alpha \neq \beta$). To avoid this contradiction we should simplify the equations of the generalized Hooke's law for the tangential stresses (11) omitting the underlined terms. It is an acceptable assumption for thin shell structures.

Indeed, consider the orthotropic layer of the shell whose axes of symmetry $\alpha_{1'}^{(k)}$, $\alpha_{2'}^{(k)}$, z do not coincide with the coordinate directions α_1, α_2, z. In axes of symmetry the equations of the generalized Hooke's law will be

$$\varepsilon_{1'1'}^{(k)} = \frac{1}{E_1^{(k)}}\sigma_{1'1'}^{(k)} - \frac{\nu_{21}^{(k)}}{E_2^{(k)}}\sigma_{2'2'}^{(k)} - \frac{\nu_{31}^{(k)}}{E_3^{(k)}}\sigma_{33}^{(k)},$$

$$\varepsilon_{2'2'}^{(k)} = -\frac{\nu_{12}^{(k)}}{E_1^{(k)}}\sigma_{1'1'}^{(k)} + \frac{1}{E_2^{(k)}}\sigma_{2'2'}^{(k)} - \frac{\nu_{32}^{(k)}}{E_3^{(k)}}\sigma_{33}^{(k)}, \tag{21}$$

$$\varepsilon_{33}^{(k)} = -\frac{\nu_{13}^{(k)}}{E_1^{(k)}}\sigma_{1'1'}^{(k)} - \frac{\nu_{23}^{(k)}}{E_2^{(k)}}\sigma_{2'2'}^{(k)} + \frac{1}{E_3^{(k)}}\sigma_{33}^{(k)}, \tag{22}$$

$$\varepsilon_{2'3}^{(k)} = \frac{1}{G_{23}^{(k)}}\sigma_{2'3}^{(k)}, \quad \varepsilon_{1'3}^{(k)} = \frac{1}{G_{13}^{(k)}}\sigma_{1'3}^{(k)}, \quad \varepsilon_{1'2'}^{(k)} = \frac{1}{G_{12}^{(k)}}\sigma_{1'2'}^{(k)}, \tag{23}$$

where $E_1^{(k)}$, $E_2^{(k)}$ and $E_3^{(k)}$ are the elastic moduli in the $\alpha_{1'}^{(k)}$, $\alpha_{2'}^{(k)}$ and z directions; $G_{12}^{(k)}$, $G_{13}^{(k)}$ and $G_{23}^{(k)}$ are the shear moduli. From reasons of symmetry we have $\nu_{\alpha\beta}^{(k)}/E_\alpha^{(k)} = \nu_{\beta\alpha}^{(k)}/E_\beta^{(k)}$ ($\alpha \neq \beta$).

As a shell is thin, with an exactitude acceptable to engineering calculations it is possible to accept the following assumption $\sigma_{33}^{(k)} \ll \sigma_{1'1'}^{(k)}, \sigma_{2'2'}^{(k)}$. Neglecting the transverse normal stress in Eq. (21) and solving for the tangential stresses, we find

$$\sigma_{1'1'}^{(k)} = c_{11}^{(k)}\,\varepsilon_{1'1'}^{(k)} + c_{12}^{(k)}\varepsilon_{2'2'}^{(k)},$$

$$\sigma_{2'2'}^{(k)} = c_{12}^{(k)}\,\varepsilon_{1'1'}^{(k)} + c_{22}^{(k)}\varepsilon_{2'2'}^{(k)},$$

$$c_{11}^{(k)} = \frac{E_1^{(k)}}{1 - \nu_{12}^{(k)}\nu_{21}^{(k)}},$$

$$c_{22}^{(k)} = \frac{E_2^{(k)}}{1 - \nu_{12}^{(k)}\nu_{21}^{(k)}},$$

$$c_{12}^{(k)} = \frac{\nu_{12}^{(k)}E_2^{(k)}}{1 - \nu_{12}^{(k)}\nu_{21}^{(k)}}. \tag{24}$$

Substituting the tangential stresses $\sigma_{1'1'}^{(k)}$ and $\sigma_{2'2'}^{(k)}$ into Eq. (22) and solving for the transverse normal stress, we obtain

$$\sigma_{33}^{(k)} = c_{13}^{(k)}\varepsilon_{1'1'}^{(k)} + c_{23}^{(k)}\varepsilon_{2'2'}^{(k)} + c_{33}^{(k)}\varepsilon_{33}^{(k)},$$

$$c_{13}^{(k)} = \frac{\nu_{13}^{(k)} + \nu_{12}^{(k)}\nu_{23}^{(k)}}{1 - \nu_{12}^{(k)}\nu_{21}^{(k)}}E_3^{(k)}, \quad c_{23}^{(k)} = \frac{\nu_{23}^{(k)} + \nu_{21}^{(k)}\nu_{13}^{(k)}}{1 - \nu_{12}^{(k)}\nu_{21}^{(k)}}E_3^{(k)},$$

$$c_{33}^{(k)} = E_3^{(k)}. \tag{25}$$

In the coordinates α_1, α_2, z the generalized Hooke's law (23)–(25) can be represented in the following form:

$$\sigma_{11}^{(k)} = b_{11}^{(k)}\varepsilon_{11}^{(k)} + b_{12}^{(k)}\varepsilon_{22}^{(k)} + b_{16}^{(k)}\varepsilon_{12}^{(k)},$$

$$\sigma_{22}^{(k)} = b_{12}^{(k)}\varepsilon_{11}^{(k)} + b_{22}^{(k)}\varepsilon_{22}^{(k)} + b_{26}^{(k)}\varepsilon_{12}^{(k)},$$

$$\sigma_{12}^{(k)} = b_{16}^{(k)}\varepsilon_{11}^{(k)} + b_{26}^{(k)}\varepsilon_{22}^{(k)} + b_{66}^{(k)}\varepsilon_{12}^{(k)}. \tag{26}$$

The remaining equations of the generalized Hooke's law are given by formulas (11). The components of the stiffness matrix $b_{\ell m}^{(k)}$ in the new axes can be found, for example, in [22].

It is a well-known fact that in the Timoshenko–Mindlin-type shell theory the equations of the generalized Hooke's law for the transverse shear and normal stresses are not satisfied pointwise, but can be satisfied in an integral sense. Therefore, according to formulas (11), (19) the following integral equations must be fulfilled:

$$\int_{\delta_{k-1}}^{\delta_k} (\sigma_{33}^{(k)} - b_{13}^{(k)}\varepsilon_{11}^{(k)} - b_{23}^{(k)}\varepsilon_{22}^{(k)} - b_{33}^{(k)}\varepsilon_{33}^{(k)} - b_{36}^{(k)}\varepsilon_{12}^{(k)})\,\mathrm{d}z = 0,$$

$$\int_{\delta_{k-1}}^{\delta_k} (\sigma_{13}^{(k)} - b_{45}^{(k)}\varepsilon_{23}^{(k)} - b_{55}^{(k)}\varepsilon_{13}^{(k)})N_k^{\pm}(z)\,\mathrm{d}z = 0$$

$$(1 \Leftrightarrow 2 \;\&\; 4 \Leftrightarrow 5). \tag{27}$$

With the help of the constitutive equations (26), (27), strain–displacement relationships (16) and expressions (19) we can obtain the constitutive equations for the stress resultants of the non-linear first-order discrete-layer theory of initially stressed anisotropic shells. However, due to their intricacy, these will not be displayed here.

Now, we have an opportunity to satisfy pointwise the equations of the three-dimensional elasticity theory (6) exactly for a plate and approximately for a shell with an exactitude acceptable for thin shell structures. Integrating Eq. (6) across the shell thickness from δ_0 to z and using the boundary conditions on the bottom surface (8) and equilibrium conditions at the layer interfaces (9), one can obtain the equations for the generalized transverse stresses

$$s_{13}^{(k)} = p_1^- - \frac{1}{A_1}\frac{\partial Q_{11}^{(k)}}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial Q_{12}^{(k)}}{\partial \alpha_2} - B_1(Q_{11}^{(k)} - Q_{22}^{(k)}) - 2B_2 Q_{12}^{(k)} - k_1 R_{13}^{(k)} \quad (1 \Leftrightarrow 2),$$

$$s_{33}^{(k)} = p_3^- - \frac{1}{A_1}\frac{\partial R_{13}^{(k)}}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial R_{23}^{(k)}}{\partial \alpha_2} - B_1 R_{13}^{(k)} - B_2 R_{23}^{(k)} + k_1 Q_{11}^{(k)} + k_2 Q_{22}^{(k)}, \quad (28)$$

where $Q_{ij}^{(k)}$ and $R_{i3}^{(k)}$ are the new stress resultants depending on the transverse coordinate:

$$Q_{ij}^{(k)} = \sum_{n=1}^{k-1} \int_{\delta_{n-1}}^{\delta_n} \sigma_{ij}^{(n)}\,\mathrm{d}z + \int_{\delta_{k-1}}^{z} \sigma_{ij}^{(k)}\,\mathrm{d}z,$$

$$R_{i3}^{(k)} = \sum_{n=1}^{k-1} \int_{\delta_{n-1}}^{\delta_n} \Sigma_{i3}^{(n)}\,\mathrm{d}z + \int_{\delta_{k-1}}^{z} \Sigma_{i3}^{(k)}\,\mathrm{d}z. \quad (29)$$

It is important to note that from Eqs. (18), (20), (28), (29) it is apparent that the boundary conditions for the generalized transverse stresses on the top shell surface (7) are also satisfied since $Q_{ij}^{(N)}(\delta_N) = \sum_{k=1}^{N}(H_{ij}^{(k)-} + H_{ij}^{(k)+})$ and $R_{i3}^{(N)}(\delta_N) = \sum_{k=1}^{N}(S_{i3}^{(k)-} + S_{i3}^{(k)+})$.

Finally, from the last formula (19) we can find the transverse stresses

$$\sigma_{i3}^{(k)} = s_{i3}^{(k)} - \beta_i^{(k)}(\tilde{\sigma}_{33}^{(k)} + s_{33}^{(k)}), \quad \sigma_{33}^{(k)} = s_{33}^{(k)}. \quad (30)$$

So, all governing relationships of the refined non-linear first-order discrete-layer theory of prestressed thin anisotropic shells have been derived.

4. Axisymmetric deformation of initially stressed multilayered shells of revolution

Let us consider the prestressed multilayered anisotropic shell of revolution with uniform circumferential properties subjected to axisymmetric loading. It is assumed that the initial stresses $\tilde{\sigma}_{\alpha\beta}^{(k)}$ are independent on the circumferential coordinate. In this case, the shell will deform axisymmetrically remaining as a body of revolution, and the displacements of the face shell surfaces and layer interfaces $v_\alpha^{(\ell)}$ $(\ell = \overline{0, N})$ will depend only on the meridional coordinate s.

Let $\mathbf{Y}$ be the state vector whose components are defined as

$$Y_1 = H_{11}^{(1)-}, \quad Y_{1+n} = H_{11}^{(n)+} + H_{11}^{(n+1)-},$$
$$Y_{N+1} = H_{11}^{(N)+}, \quad Y_{N+2} = H_{12}^{(1)-},$$
$$Y_{N+2+n} = H_{12}^{(n)+} + H_{12}^{(n+1)-},$$
$$Y_{2N+2} = H_{12}^{(N)+}, \quad Y_{2N+3} = S_{13}^{(1)-},$$
$$Y_{2N+3+n} = S_{13}^{(n)+} + S_{13}^{(n+1)-}, \quad Y_{3N+3} = S_{13}^{(N)+},$$
$$Y_{3N+4+\ell} = v_1^{(\ell)},$$
$$Y_{4N+5+\ell} = v_2^{(\ell)}, \quad Y_{5N+6+\ell} = v_3^{(\ell)}$$
$$(n = \overline{1, N-1};\ \ell = \overline{0, N}). \quad (31)$$

Taking into account relationships (17), (18), (20), (31) and constitutive equations we can write the governing system of non-linear differential equations in the following vector form:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}s} = \mathbf{F}(s, \mathbf{Y}). \quad (32)$$

The boundary conditions of the axisymmetric deformation problem according to formulas (14) can be written as follows:

$$Y_n(s^-)\ell_n + Y_{3N+3+n}(s^-)(1 - \ell_n) = 0,$$
$$Y_n(s^+)\ell_{3N+3+n} + Y_{3N+3+n}(s^+)(1 - \ell_{3N+3+n}) = 0, \quad (33)$$

where Y_n and Y_{3N+3+n} are the components of the vector $\mathbf{Y}$; ℓ_n and ℓ_{3N+3+n} are the boundary coefficients, which may take the values 0 and 1, and define any homogeneous static or kinematic boundary conditions at the left edge $s = s^-$ and right edge $s = s^+$ of the shell, where $n = \overline{1, 3N+3}$.

The non-linear boundary value problem (32), (33) can be reduced to a sequence of linear boundary value problems by using the Newton–Raphson method as

$$\frac{d\mathbf{Y}^{[m+1]}}{ds} = \mathbf{A}(s, \mathbf{Y}^{[m]}) \cdot \mathbf{Y}^{[m+1]} + \mathbf{G}(s, \mathbf{Y}^{[m]}). \quad (34)$$

The linear boundary value problem (33), (34) is solved by application of the discrete orthogonalization method [3]. The process starts with $\mathbf{Y}^{[0]} = \mathbf{0}$ and we carry on it until the inequality

$$\max_{\ell} |(Y_{\ell}^{[m+1]} - Y_{\ell}^{[m]})/Y_{\ell}^{[m+1]}| < \varepsilon$$

will be satisfied for a priori chosen parameter ε, where $\ell = \overline{1, 6N+6}$.

As a numerical example, we consider a relatively simple problem of the non-linear axisymmetric response of the multilayered anisotropic tire. For the sake of simplicity, the tire is modeled as a four-layered anisotropic toroidal shell (the so-called bias-ply tire) which has a circular cross-section (see Fig. 3). The shell is subjected to uniform inflation pressure $p_3^- = -q$, where $q = 0.15$ MPa. The material characteristics of the layers are taken to be those typical of cord-rubber composites [12]: $E_L = 510.45$ MPa, $E_T = 6.91$ MPa, $G_{LT} = 2.33$ MPa, $G_{TT} = 1.77$ MPa, $\nu_{LT} = 0.46$, $\nu_{TT} = 0.95$,

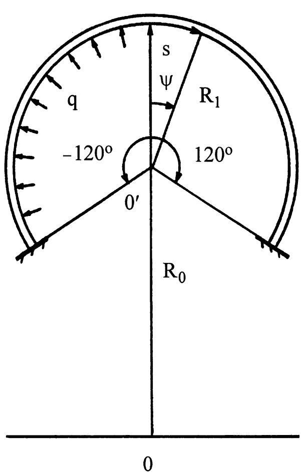

Fig. 3. Four-layered composite toroidal shell subjected to inflation pressure.

where the subscripts L and T refer to longitudinal and transverse directions of the individual ply. Let the geometrical characteristics of the inner surface of the shell are $R_1 = 50$ mm and $R_0 = 250$ mm; thicknesses of the shell and plies are $h = 4.8$ mm and $h_k = h/N$ mm; ply orientations are $\gamma_k = (-1)^{k-1}\gamma$, where $N = 4$, $\gamma = 52°$ and $k = 1, 2, 3, 4$. The tire is assumed to be rigidly clamped at the rim (at $\psi = \pm 120°$).

This non-linear problem can be also solved by using the incremental method [21]. Let the tire be loaded to 0.15 MPa inflation pressure in five load steps $q_n = 0.03n$ MPa, where $n = \overline{1, 5}$. At each of the load steps the geometrically linear problem for a prestressed shell of revolution is solved. It should be noted that the effects of the meridian stretching and thickness variation under the new geometry computation were not taken into account. Other feature of this approach is the non-conservative character of the pressure loading since the displacements are referred at each of the load steps from a new reference surface.

The numerical results presented in Fig. 4 have been obtained by using the incremental method (see the solid lines with various values of the load parameter q) and the Newton–Raphson method (see curves marked by ●). Note that only four iterations were required for finding the solution of the geometrically non-linear problem with the given accuracy $\varepsilon = 10^{-4}$. Additionally, in Fig. 4 the solution of the geometrically linear problem is given (see curves marked by ◆). The distribution of the stress components in the thickness direction is shown for the middle cross-section (at $\psi = 60°$). It is seen that both the numerical solutions of the non-linear problem lead to the similar results. Note that transverse shear stresses σ_{13} and σ_{23} obtained by using the Newton–Raphson method and incremental method do not vanish at the inner surface of the shell and are discontinuous at the layer interfaces. It can be explained by allowing for the non-linear terms in formula (30). However, this effect is appreciable only for the finite deflection problems.

As already said, two layers of this cord-rubber composite are put together with $\pm\gamma$ fiber orientations with respect to the meridional direction. Each layer separately would try to exhibit the shear

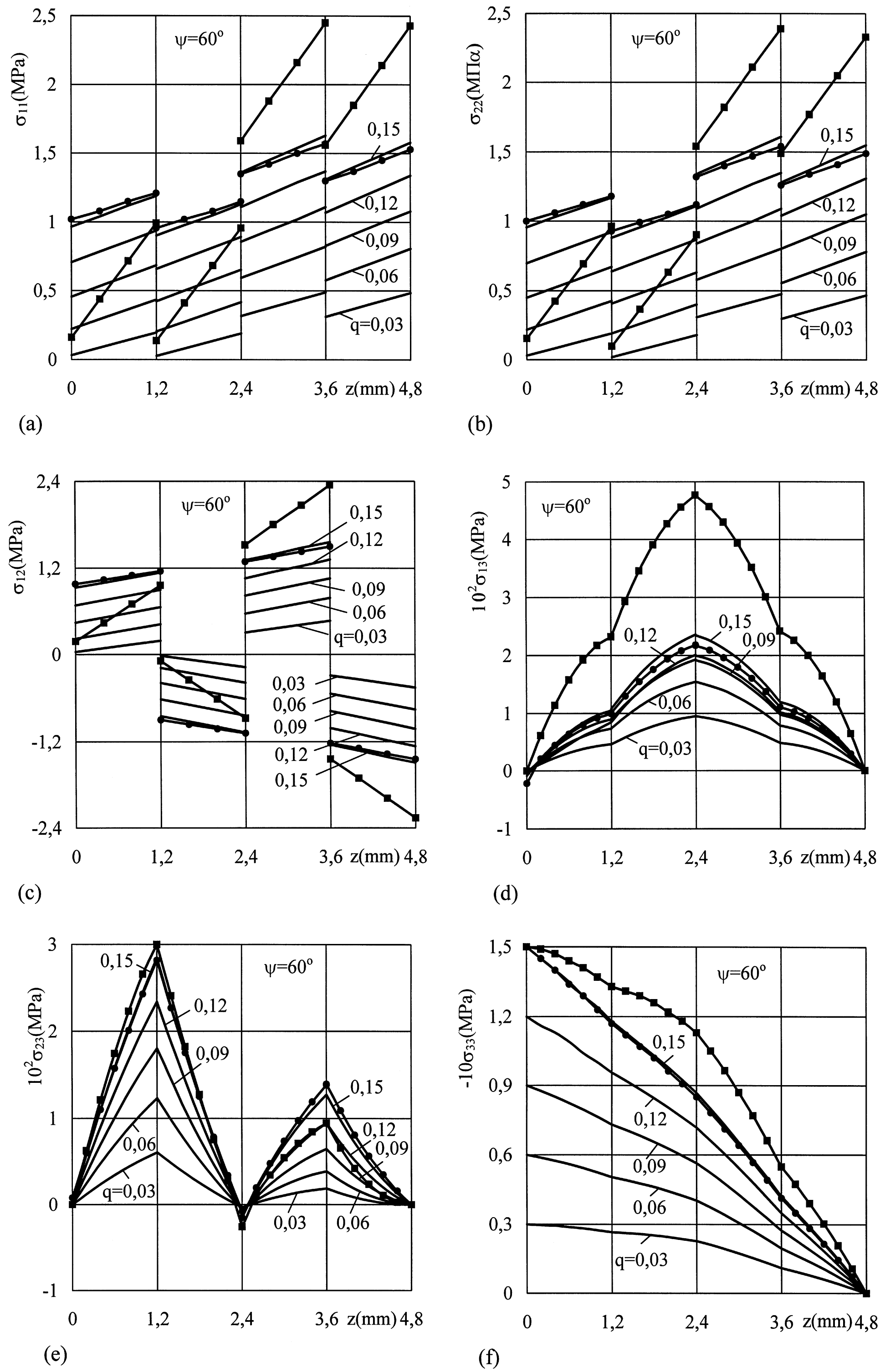

Fig. 4. The distribution of the stresses in the thickness direction at the cross-section ($\psi = 60°$): (a) σ_{11}; (b) σ_{22}; (c) σ_{12}; (d) σ_{13}; (e) σ_{23}; (f) σ_{33}. (—) Incremental method; (-●-●-) Newton–Raphson method; (-◆-◆-) linear solution.

coupling behavior. Their shearing actions would be in opposite directions due to their opposite fiber orientations. The mutual interaction between the layers would try to restrict in-plane shear motions and as a result would generate transverse shear stresses σ_{23} that are essential in pneumatic tires. It is apparent that in a case of using the traditional non-linear theory of laminated orthotropic shells we will lose this effect,[2] i.e. $\sigma_{23} = 0$. In this connection let us pay attention to the same order of the transverse shear stresses σ_{13} and σ_{23} that it is noticeable, namely, for the non-linear problem (see for a comparison Figs. 4(d) and (e)). It points to an essential influence of anisotropy and geometrical non-linearity on the stress field in bias-ply tires.

It should be mentioned, that due to the essentially non-uniform distribution of the transverse shear stresses σ_{13} and σ_{23} over the thickness of a tire, the Timoshenko–Mindlin-type shell theory does not provide the reliable prediction of tire failure.

5. Conclusions

The refined first-order discrete-layer theory of prestressed anisotropic shells has been developed. The effects of the laminated anisotropic material response, initially stressed state response, geometrical non-linearity, transverse shear and transverse normal strains are included. This theory is based on the refined Grigolyuk's hypothesis adopted for the displacement vector. As unknown functions the tangential and transverse displacements of the face surfaces of the shell and layer interfaces have been chosen. Such choice of unknowns allows as much as possible to algorithmize the computational modeling of a series of important shell problems. The governing equations of the theory of prestressed multilayered anisotropic shells have been obtained by using the principle of the virtual work and the well-known Novozhilov's partially non-linear strain–displacement relationships. The developed theory can be used for solving the shell problems where the above effects are significant. Such problems can be met in many fields of the engineering science and especially in the tire mechanics.

The computational model for solving the axisymmetric problems of prestressed multilayered anisotropic shells of revolution has been presented. This computational model is based on the Newton–Raphson method and incremental method through using the discrete orthogonalization method. For example, a relatively simple problem of the non-linear axisymmetric response of the anisotropic bias-ply tire has been solved. The tire is modeled by the four-layered cross-ply toroidal shell subjected to inflation pressure. It has been shown that both the numerical solutions give similar results, excepting the values of the transverse shear stresses at the inner surface of a tire. It has been also established that neglecting the effects of anisotropy and geometrical non-linearity can lead to an incorrect description of the stress field in bias-ply tires.

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