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MECHANICS

On the Shear Correction Factor in the Timoshenko-Type Shell Theory

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As is well known, in the theory of Timoshenko-type shells the shear correction factors k are introduced to allow for the nonuniformity of the transverse-shear distribution in shell cross sections [1]. At present, the following values are commonly used for the shear correction factors: $k = \frac{5}{6}$ [2] and $k = \frac{\pi^2}{12}$ [3]. These correction factors were obtained as a result of analysis of thin elastic plates and are consistent with one another. As is shown below, in the case of the approach [3] (in which the derivation of governing relations is based on the principle of virtual work), the value of the shear correction factor $k = 1$ is preferable. In this case, the transverse components of the stress tensor can be recovered by integrating equations of the spatial elasticity theory. The choice $k = 1$ makes it possible to construct a mathematically consistent and noncontradictory theory of Timoshenko-type shells.

1. We consider a thin anisotropic shell with a constant thickness h. It is assumed that at each shell point there exists a surface of elastic symmetry that is parallel to face surfaces S^- and S^+. We take as an reference surface S an arbitrary internal shell surface located at the distances δ^- and δ^+ from the face surfaces; i.e., $h = \delta^+ - \delta^-$. We associate the reference surface with orthogonal coordinates α_1 and α_2, which are counted off along the lines of principal curvatures. The α_3-coordinate is counted off in the direction of increasing the outer normal to the surface S.

In linear elasticity theory, the equations of equilibrium for a thin shell whose face-surface metrics can be identified with the metric of the reference surface have the form

$$\frac{1}{A_i}\frac{\partial\sigma_{ii}}{\partial\alpha_i} + \frac{1}{A_j}\frac{\partial\sigma_{ij}}{\partial\alpha_j} + \frac{\partial\sigma_{i3}}{\partial\alpha_3} + B_i(\sigma_{ii} - \sigma_{jj}) + 2B_j\sigma_{ij} + k_i\sigma_{i3} = 0, \quad i \neq j,$$
$$\frac{1}{A_1}\frac{\partial\sigma_{13}}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial\sigma_{23}}{\partial\alpha_2} + \frac{\partial\sigma_{33}}{\partial\alpha_3} + B_1\sigma_{13} + B_2\sigma_{23} - k_1\sigma_{11} - k_2\sigma_{22} = 0. \tag{1}$$

Here, $B_i = \frac{\partial A_j/\partial\alpha_i}{A_1A_2}$; $\sigma_{\alpha\beta}$ are stresses; A_i and k_i are the Lamé parameters and curvatures of the coordinate lines, respectively; $i, j = 1, 2$; and $\alpha, \beta = 1, 2, 3$.

The equations of the generalized Hooke law with allowance for the admission $\sigma_{33} \ll \sigma_{ij}$ can be written out in the form

$$\sigma_{ij} = \sum_{l \le m} b_{ijlm}\varepsilon_{lm}, \quad \sigma_{i3} = \sum_{l} b_{i3l3}\varepsilon_{l3}, \quad i, j, l, m = 1, 2. \tag{2}$$

In constructing the theory, we employ the modified Timoshenko hypothesis [4] on the linear distribution of displacements across the shell thickness

$$u_i = N^-(\alpha_3)v_i^- + N^+(\alpha_3)v_i^+, \quad u_3 = v_3,$$
$$N^-(\alpha_3) = \frac{\delta^+ - \alpha_3}{h}, \quad N^+(\alpha_3) = \frac{\alpha_3 - \delta^-}{h}, \tag{3}$$

where $v_i^{\pm}(\alpha_1, \alpha_2)$ are tangential displacements of the face surfaces $S^{\pm}$, and $v_3(\alpha_1, \alpha_2)$ is the transverse displacement of the surface S.

We now introduce displacements (3) into the strain displacement relations of the linear elasticity theory. Then, assuming the transverse shears to be distributed uniformly across the shell thickness, we arrive at the expressions

$$\varepsilon_{ij} = N^-(\alpha_3)e_{ij}^- + N^+(\alpha_3)e_{ij}^+, \quad \varepsilon_{i3} = e_{i3}, \quad \varepsilon_{33} = 0,$$

$$e_{ii}^{\pm} = \frac{1}{A_i}\frac{\partial v_i^{\pm}}{\partial\alpha_i} + B_j v_j^{\pm} + k_i v_3 \quad i \neq j,$$

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$$e_{12}^{\pm} = \frac{1}{A_1}\frac{\partial v_2^{\pm}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial v_1^{\pm}}{\partial \alpha_2} - B_2 v_1^{\pm} - B_1 v_2^{\pm}, \quad (4)$$

$$e_{i3} = \beta_i - \theta_i, \quad \beta_i = \frac{1}{h}(v_i^{+} - v_i^{-}),$$

$$\theta_i = k_i v_i - \frac{1}{A_i}\frac{\partial v_3}{\partial \alpha_i}, \quad v_i = \frac{1}{2}(v_i^{-} + v_i^{+}).$$

We multiply first two equations of equilibrium (1) by the shape functions $N^{\pm}(\alpha_3)$ and integrate them (together with the third equation) with respect to the transverse coordinate within the limits from δ^- to δ^+, with the boundary conditions $\sigma_{\alpha 3}(\delta^{\pm}) = p_{\alpha}^{\pm}$ taken into account. As a result, we arrive at the equations of equilibrium for a shell with respect to the stress resultants

$$\frac{1}{A_i}\frac{\partial H_{ii}^{\pm}}{\partial \alpha_i} + \frac{1}{A_j}\frac{\partial H_{ij}^{\pm}}{\partial \alpha_j} + B_i(H_{ii}^{\pm} - H_{jj}^{\pm})$$

$$+ 2B_j H_{ij}^{\pm} + \left(\frac{1}{2}k_i \mp \frac{1}{h}\right)T_{i3} = \mp p_i^{\pm}, \quad i \neq j,$$

$$\frac{1}{A_1}\frac{\partial T_{13}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial T_{23}}{\partial \alpha_2} + B_1 T_{13} + B_2 T_{23} \quad (5)$$

$$- k_1 T_{11} - k_2 T_{22} = p_3^{-} - p_3^{+},$$

$$H_{ij}^{\pm} = \int_{\delta^-}^{\delta^+} \sigma_{ij} N^{\pm}(\alpha_3) d\alpha_3, \quad T_{i\alpha} = \int_{\delta^-}^{\delta^+} \sigma_{i\alpha} d\alpha_3.$$

Here, $T_{i\alpha}$ and $H_{ij}^{\pm}$ are the classical and generalized stress resultants, respectively, and $p_{\alpha}^{\pm}$ are the surface loads acting on the face surfaces $S^{\pm}$.

With allowance for relationships (2), the constituting equations for specific forces and moments can be represented in the form

$$H_{ij}^{-} = \frac{1}{6}h \sum_{l \le m} b_{ijlm}(2e_{lm}^{-} + e_{lm}^{+}),$$

$$H_{ij}^{+} = \frac{1}{6}h \sum_{l \le m} b_{ijlm}(e_{lm}^{-} + 2e_{lm}^{+}), \quad (6)$$

$$T_{ij} = H_{ij}^{-} + H_{ij}^{+}, \quad T_{i3} = h \sum_{l} k_{il} b_{i3l3} e_{l3},$$

where we assumed $k_{il} = 1$ for the shear correction factors. We should note that formula (6) for the transverse forces T_{i3} expresses a rather simple fact. This formula implies that the elasticity relations for the transverse shear stresses (2) in the Timoshenko-type shell theory are not satisfied pointwise but are fulfilled as integral relations across the shell thickness [4, 5].

Furthermore, we integrate Eqs. (1) of the spatial elasticity theory over the transverse coordinate from δ^- to α_3. Taking into account the boundary conditions $\sigma_{\alpha 3}(\delta^-) = p_{\alpha}^{-}$, we arrive at the formulas for the determination of the stress transverse components

$$\sigma_{i3} = p_i^{-} - \frac{1}{A_i}\frac{\partial Q_{ii}}{\partial \alpha_i} - \frac{1}{A_j}\frac{\partial Q_{ij}}{\partial \alpha_j}$$

$$- B_i(Q_{ii} - Q_{jj}) - 2B_j Q_{ij} - k_i Q_{i3}, \quad i \neq j, \quad (7)$$

$$\sigma_{33} = p_3^{-} - \frac{1}{A_1}\frac{\partial Q_{13}}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial Q_{23}}{\partial \alpha_2}$$

$$- B_1 Q_{13} - B_2 Q_{23} + k_1 Q_{11} + k_2 Q_{22},$$

$$Q_{i\alpha} = \int_{\delta^-}^{\alpha_3} \sigma_{i\alpha} d\alpha_3. \quad (8)$$

We pay attention to the fact that, by virtue of the equality $Q_{i\alpha}(\delta^+) = T_{i\alpha}$ and equations of equilibrium (5), the boundary conditions $\sigma_{\alpha 3}(\delta^+) = p_{\alpha}^{+}$ immediately follow from relationships (7).

2. We now discuss a statement important for the theory of Timoshenko-type shells and associated with the validity of equations of equilibrium (5) for a shell in the case of the stress field (2), (7), which was found as a result of solving the problem. The matter is that upon the determination of the transverse shear stresses σ_{i3} according to formula (7) and the calculation on this basis of the transverse forces T_{i3}^{s}, we can encounter a situation when listed equations of equilibrium (5) for a shell are not exactly satisfied. The reason consists in the fact that the transverse forces T_{i3}, whose calculation is based on Hooke law (2), i.e., on formula (6), in the general case, can be not coincident with T_{i3}^{s}.

In order to solve the problem posed, we employ the formulas following from relationships (2), (4), and (8):

$$\int_{\delta^-}^{\delta^+} Q_{ij} d\alpha_3 = hH_{ij}^{-}, \quad \int_{\delta^-}^{\delta^+} Q_{i3} d\alpha_3 = \frac{1}{2}hT_{i3}.$$

With allowance for these formulas, as well as for relation (7), we obtain the expression

$$T_{i3}^{s} = \int_{\delta^-}^{\delta^+} \sigma_{i3} d\alpha_3 = h\Bigg[p_i^{-} - \frac{1}{A_i}\frac{\partial H_{ii}^{-}}{\partial \alpha_i} - \frac{1}{A_j}\frac{\partial H_{ij}^{-}}{\partial \alpha_j}$$

$$- B_i(H_{ii}^{-} - H_{jj}^{-}) - 2B_j H_{ij}^{-} - \frac{1}{2}k_i T_{i3}\Bigg], \quad i \neq j. \quad (9)$$

With equations of equilibrium (5) for a shell taken into account, we have from Eq. (9) that $T_{i3}^{s} = T_{i3}$, which was in need of proof.

In conclusion, it is worth emphasizing that we managed to construct the noncontradictory theory for Timoshenko-type shells [in the sense of the simultaneous satisfaction of the equations of equilibrium for a shell (5) and relationships (7)]. Such a construction became possible on the basis of a physically clear assumption about the integral validity of equations corresponding to the Hooke law for transverse shear stresses (2). This implies that we should admit $k_{il} = 1$ in formula (6) for transverse forces. In this connection, we note that, from the standpoint of the approach developed in this paper, attempts to construct theories for Timoshenko-type shells, which are based on one concept or another related to calculation methods for the shear correction factors [1], will result in mathematically inconsistent and contradictory theories.

REFERENCES

1. É. I. Grigolyuk and I. T. Selezov, *Mechanics of Deformable Solids*, Vol. 5: *Nonclassical Theories of Oscillations of Bars, Plates and Shells* (VINITI, Moscow, 1973).
2. E. Reissner, J. Math. Phys. **23** (4), 184 (1944).
3. R. D. Mindlin, J. Appl. Mech. **18** (1), 31 (1951).
4. G. M. Kulikov, J. Eng. Mech. **127** (2), 119 (2001).
5. É. I. Grigolyuk and G. M. Kulikov, *Multi-Layer Reinforced Shells: Analysis of Pneumatic Tires* (Mashinostroenie, Moscow, 1988).

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