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THE CONTACT PROBLEM FOR A GEOMETRICALLY NON-LINEAR TIMOSHENKO-TYPE SHELL†

G. M. KULIKOV and S. V. PLOTNIKOVA

Tambov

e-mail: kulikov@apmath.tstu.ru

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An algorithm is developed for the numerical solution of the contact problem of an elastic Timoshenko-type shell subjected to arbitrarily large displacements and rotations, using mixed finite-element approximations. It is essential that six displacements of the faces of the shell are chosen as the required functions. This enables one, first, to simplify the formulation of contact problems in the mechanics of thin-walled structures, since functions by means of which the conditions for the non-penetration of the bodies are formulated are chosen as the required functions and, second, to obtain relations for the components of the Green–Lagrange strain tensor in curvilinear, orthogonal coordinates which accurately represent arbitrarily large displacements of a shell as a rigid body.

The finite-element method is the most powerful numerical method for solving contact problems in the mechanics of shells. At the same time, the problem of constructing twisted finite elements of thin shells which have been subjected to arbitrarily large rotations and are interacting with rigid bodies is still far from being solved [1, 2]. The reason is concealed in the inadequate representation of large displacements of an element of a shell as a rigid whole by the deformation relations. It is therefore not surprising that, since there are no deformation relations in the literature which are capable of exactly representing an arbitrarily deformed state of a shell in local curvilinear coordinates, the concept of a "degenerate" (isoparametric) element has been essentially developed [3], which enables one, at the price of a substantial increase in the time required for the calculation, to represent arbitrarily large displacements of an element of a shell as a rigid whole in global Cartesian system of coordinates [2, 4].

An algorithm for the numerical solution of the contact problem for a shell which is subjected to arbitrarily large displacements and rotations is developed below using mixed finite-element approximations [7, 8] on the basis of the theory of elastic Timoshenko-type shells taking transverse compression into account [5, 6]. Since the displacements of the faces of the shell are chosen as the required functions [9, 10], this simplifies the formulation of contact problems in the mechanics of thin-walled structures [11].

1. DEFORMATION RELATIONS IN THE GEOMETRICALLY NON-LINEAR THEORY OF TIMOSHENKO-TYPE SHELLS

Consider a shell of constant thickness h. As the reference surface S, we will take the internal surface of the shell, which we will refer to the orthogonal curvilinear coordinates α_1 and α_2, which are measured along the lines of principal curvatures. We shall read off the transverse coordinate α_3 in the direction of the outward normal to the surface S (Fig. 1). Suppose $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors which are tangent to the coordinates lines α_1 and α_2, $\mathbf{e}_3$ is the unit vector of the normal, A_α and k_α are the Lamé parameters and curvatures of the coordinate lines of the surface S, δ^- and δ^+ are the distances from the surface S to the lower surface S^- and upper surface S^+ of the shell, u_i are the components of the displacement vector, ε_{ij} are the components of the Green–Lagrange strain tensor, $p_i^{\mp}$ are the external surface loads acting on the faces $S^{\pm}$, $\mathbf{q} = q_\nu\boldsymbol{\nu} + q_t\mathbf{t} + q_3\mathbf{e}_3$ is the vector of the external surface loads acting on the lateral surface $\boldsymbol{\Omega}$ and $\boldsymbol{\nu}$ and $\mathbf{t}$ are the normal and tangential unit vectors to the boundary contour Γ (Fig. 1). Here and henceforth, the Latin subscripts $i, j, l, m = 1, 2, 3$ and the Greek subscripts $\alpha, \beta, \gamma, \delta = 1, 2$.

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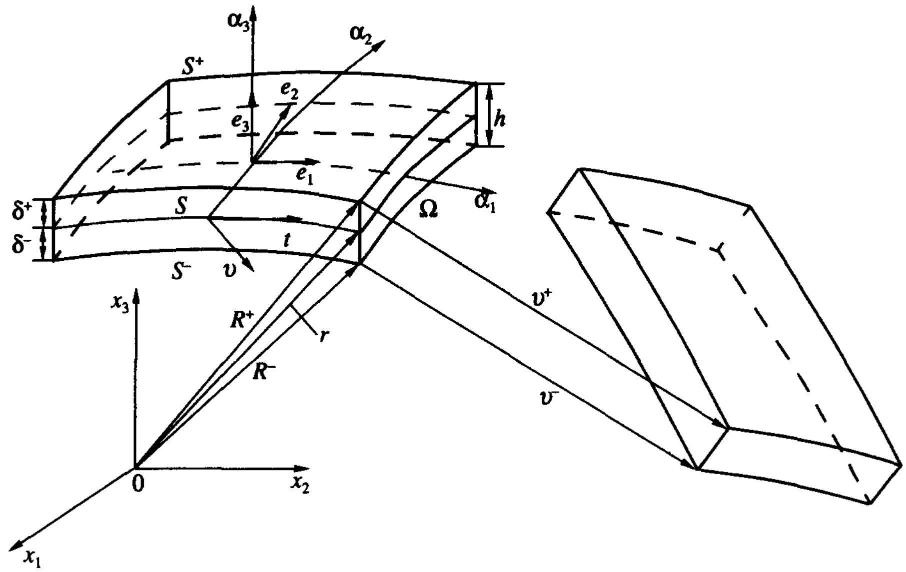

Fig. 1

We shall make use of Timoshenko's kinematic hypothesis concerning the linear distribution of the displacements throughout the thickness of the shell [5, 6]

$$\mathbf{u} = \sum_{\pm} N^{\pm}(\alpha_3)\mathbf{v}^{\pm}, \quad \mathbf{u} = \sum_i u_i \mathbf{e}_i, \quad \mathbf{v}^{\pm} = \sum_i v_i^{\pm}\mathbf{e}_i \tag{1.1}$$

where $\mathbf{v}^{\pm}$ are the displacement vectors of the points of the faces $S^{\pm}$, $v_i^{\pm}(\alpha_1, \alpha_2)$ are the components of these vectors and $N^{\pm}(\alpha_3)$ are linear functions of the form of the shell

$$N^{-}(\alpha_3) = \frac{1}{h}(\delta^{+} - \alpha_3), \quad N^{+}(\alpha_3) = \frac{1}{h}(\alpha_3 - \delta^{-}) \tag{1.2}$$

We now introduce the displacements (1.1) into the strain relations of the three-dimensional theory of elasticity 1 [12] and, assuming that the tangential components of the Green–Lagrange strain tensor vary linearly along the thickness of the shell, we arrive at the strain relations of the geometrically nonlinear theory of Timoshenko-type shells of average thickness [13]

$$\begin{aligned}
2\varepsilon_{\alpha\beta} &= \sum_{\pm} N^{\pm}(\alpha_3)\left(\frac{1}{H_\alpha^{\pm}}\mathbf{v}^{\pm}_{,\alpha}\cdot\mathbf{e}_\beta + \frac{1}{H_\beta^{\pm}}\mathbf{v}^{\pm}_{,\beta}\cdot\mathbf{e}_\alpha + \frac{1}{H_\alpha^{\pm}H_\beta^{\pm}}\mathbf{v}^{\pm}_{,\alpha}\cdot\mathbf{v}^{\pm}_{,\beta}\right)\\
2\varepsilon_{\alpha 3} &= \boldsymbol{\beta}\cdot\mathbf{e}_\alpha + \frac{1}{\bar{H}_\alpha}\bar{\mathbf{v}}_{,\alpha}\cdot(\mathbf{e}_3 + \boldsymbol{\beta}) + (\alpha_3 - \bar{\delta})\frac{1}{\bar{H}_\alpha}\varepsilon_{33,\alpha}, \quad \varepsilon_{33} = \boldsymbol{\beta}\cdot\left(\mathbf{e}_3 + \frac{1}{2}\boldsymbol{\beta}\right)\\
\boldsymbol{\beta} &= \frac{1}{h}(\mathbf{v}^{+} - \mathbf{v}^{-}), \quad \bar{\mathbf{v}} = \frac{1}{2}(\mathbf{v}^{-} + \mathbf{v}^{+})
\end{aligned} \tag{1.3}$$

where $H_\alpha^{\pm} = A_\alpha(1 + k_\alpha\delta^{\pm})$ are the Lamé parameters of the faces $S^{\pm}$, $\bar{H}_\alpha = A_\alpha(1 + k_\alpha\bar{\delta})$ are the Lamé parameters of the middle surface $\bar{S}$, $\bar{\delta} = (\delta^{-} + \delta^{+})/2$ is the distance from the reference surface to the middle surface, $\bar{\mathbf{v}}$ is the vector of the displacement of a point of the middle surface of the shell, and the subscript β, which follows after a comma, denotes partial differentiation with respect to α_β.

The strain relations (1.3) are extremely attractive from the point of view of their use in the finite-element method, since they *exactly* represent arbitrarily large displacements of a shell as a rigid body.

Actually, the displacements of the faces of a shell as a rigid whole [14] can be represented in the form

$$\mathbf{v}^{\pm R} = \boldsymbol{\Delta} + \boldsymbol{\Phi}\mathbf{R}^{\pm} - \mathbf{R}^{\pm} \tag{1.4}$$

where $\mathbf{R}^{\pm} = \mathbf{r} + \delta^{\pm}\mathbf{e}_3$ are the radius vectors of the points of the surfaces, $\mathbf{r}$ is the radius vector of a point of the reference surface, $\mathbf{\Delta}$ is the vector of the translational displacements of the shell and $\mathbf{\Phi}$ is an orthogonal matrix which characterizes the rotation of the shell as a rigid whole about the point O (Fig. 1). The formula

$$\frac{1}{H_\alpha^{\pm}}\mathbf{v}_{,\alpha}^{\pm R} = \mathbf{\Phi}\mathbf{e}_\alpha - \mathbf{e}_\alpha \tag{1.5}$$

holds for the derivates of the rigid displacement vectors (1.4). On then introducing expressions (1.4) and (1.5) into the strain relations (1.3) and taking into account the property of an orthogonal transform to preserve the scalar vector product, we obtain

$$2\varepsilon_{\alpha\beta}^R = (\mathbf{\Phi}\mathbf{e}_\alpha)\cdot(\mathbf{\Phi}\mathbf{e}_\beta) - \mathbf{e}_\alpha\cdot\mathbf{e}_\beta = 0,\quad 2\varepsilon_{33}^R = (\mathbf{\Phi}\mathbf{e}_3)\cdot(\mathbf{\Phi}\mathbf{e}_3) - \mathbf{e}_3\cdot\mathbf{e}_3 = 0$$

$$2\varepsilon_{\alpha 3}^R = \frac{1+k_\alpha\alpha_3}{1+k_\alpha\bar{\delta}}[(\mathbf{\Phi}\mathbf{e}_\alpha)\cdot(\mathbf{\Phi}\mathbf{e}_3) - \mathbf{e}_\alpha\cdot\mathbf{e}_3] = 0$$

which it was required to prove.

For the purpose of using the strain relations (1.3) in the algorithm for the numerical solution of contact problems, we will represent them in the scalar form

$$\varepsilon_{\alpha i} = \sum_{\pm} N^{\pm}(\alpha_3)(e_{\alpha i}^{\pm} + \eta_{\alpha i}^{\pm}),\quad \varepsilon_{33} = e_{33} + \eta_{33} \tag{1.6}$$

where

$$\varepsilon_{\alpha\alpha}^{\pm} = \frac{1}{\zeta_\alpha^{\pm}}\lambda_\alpha^{\pm},\quad 2e_{12}^{\pm} = \frac{1}{\zeta_1^{\pm}}\omega_1^{\pm} + \frac{1}{\zeta_2^{\pm}}\omega_2^{\pm}$$

$$2e_{\alpha 3}^{\pm} = \left(1 \pm \frac{k_\alpha h}{2\bar{\zeta}_\alpha}\right)\beta_\alpha - \frac{1}{\bar{\zeta}_\alpha}\theta_\alpha^{\pm},\quad e_{33} = \beta_3$$

$$\eta_{\alpha\alpha}^{\pm} = \frac{1}{2(\zeta_\alpha^{\pm})^2}[(\lambda_\alpha^{\pm})^2 + (\omega_\alpha^{\pm})^2 + (\theta_\alpha^{\pm})^2],\quad \eta_{12}^{\pm} = \frac{1}{2\zeta_1^{\pm}\zeta_2^{\pm}}(\lambda_1^{\pm}\omega_2^{\pm} + \lambda_2^{\pm}\omega_1^{\pm} + \theta_1^{\pm}\theta_2^{\pm})$$

$$\eta_{\alpha 3}^{\pm} = \frac{1}{2\bar{\zeta}_\alpha}(\beta_\alpha\lambda_\alpha^{\pm} + \beta_\gamma\omega_\alpha^{\pm} - \beta_3\theta_\alpha^{\pm}),\quad \eta_{33} = \frac{1}{2}(\beta_1^2 + \beta_2^2 + \beta_3^2) \tag{1.7}$$

$$\lambda_\alpha^{\pm} = \frac{1}{A_\alpha}v_{\alpha,\alpha}^{\pm} + B_\gamma v_\gamma^{\pm} + k_\alpha v_3^{\pm},\quad \omega_\alpha^{\pm} = \frac{1}{A_\alpha}v_{\gamma,\alpha}^{\pm} - B_\gamma v_\alpha^{\pm}$$

$$\theta_\alpha^{\pm} = -\frac{1}{A_\alpha}v_{3,\alpha}^{\pm} + k_\alpha v_\alpha^{\pm},\quad \beta_i = \frac{1}{h}(v_i^{+} - v_i^{-})$$

$$\zeta_\alpha^{\pm} = 1 + k_\alpha\delta^{\pm},\quad \bar{\zeta}_\alpha = 1 + k_\alpha\bar{\delta},\quad B_\alpha = \frac{1}{A_1A_2}A_{\gamma,\alpha}\quad (\gamma \neq \alpha)$$

2. THE HU–WASHIZU FUNCTIONAL FOR A GEOMETRICALLY NON-LINEAR TIMOSHENKO-TYPE SHELL

It is well known that the equilibrium equations, the strain relations, the generalized Hooke's law equations and the boundary conditions on the face and end surfaces of a shell constitute the Euler equations and the natural boundary conditions of a certain variational problem. In this connection we introduce the approximations of the displacements (1.1) and the *independently* introduced deformations.

$$\varepsilon_{\alpha i} = \sum_{\pm} N^{\pm}(\alpha_3) E^{\pm}_{\alpha i}, \quad \varepsilon_{33} = E_{33} \tag{2.1}$$

where $E^{\pm}_{\alpha\beta}(\alpha_1, \alpha_2)$ and $E^{\pm}_{\alpha 3}(\alpha_1, \alpha_2)$ are the tangential and transverse tangential deformations of the faces of the shell and $E_{33}(\alpha_1, \alpha_2)$ is the transverse compression of the shell, into the Hu–Washizu functional of the three-dimensional theory of elasticity [15]. As a result, when relations (1.6) and (1.7) are taken into account, we arrive at the following formula for the Hu–Washizu functional

$$\begin{gathered} J_{HW} = \iint_S \left\{ \Pi - \sum_{\pm} \left[\sum_{i+j<6} T^{\pm}_{ij}(E^{\pm}_{ij} - e^{\pm}_{ij} - \eta^{\pm}_{ij}) + \sum_i (\pm p^{\pm}_i) v^{\pm}_i \right] - T_{33}(E_{33} - e_{33} - \eta_{33}) \right\} \times \\ \times A_1 A_2 \bar{\zeta}_1 \bar{\zeta}_2 d\alpha_1 d\alpha_2 - \oint_{\Gamma} \sum_{\pm} (\hat{T}^{\pm}_{\nu\nu} v^{\pm}_{\nu} + \hat{T}^{\pm}_{\nu t} v^{\pm}_t + \hat{T}^{\pm}_{\nu 3} v^{\pm}_3)(1 + k_N \bar{\delta}) ds \end{gathered} \tag{2.2}$$

Here, $v^{\pm}_{\nu}$, $v^{\pm}_t$, $v^{\pm}_3$ are the components of the displacement vectors of the faces $S^{\pm}$ in the local basis $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{e}_3$ (Fig. 1), k_N is the normal curvature of the curve Γ, $\Pi(E^{\pm}_{\alpha i}, E_{33})$ is the elastic potential of the shell, $T^{\pm}_{\alpha i}$, T_{33} are the resulting stresses and $\hat{T}^{\pm}_{\nu\nu}$, $\hat{T}^{\pm}_{\nu t}$, $\hat{T}^{\pm}_{\nu 3}$ are the resulting external surface loads which are determined using the formulae

$$\Pi = \frac{1}{2} \sum_{i,j} \sum_{l,m} [D^{0-}_{ijlm} E^{-}_{ij} E^{-}_{lm} + D^{1-}_{ijlm} E^{-}_{ij} E^{+}_{lm} + D^{0+}_{ijlm} E^{+}_{ij} E^{-}_{lm} + D^{1+}_{ijlm} E^{+}_{ij} E^{+}_{lm}] \tag{2.3}$$

$$\begin{gathered} D^{n\pm}_{ijlm} = \int_{\delta^-}^{\delta^+} b_{ijlm} N^{\pm}(\alpha_3) [N^{-}(\alpha_3)]^{1-n} [N^{+}(\alpha_3)]^{n} d\alpha_3, \quad n = 0, 1 \\ T^{\pm}_{\alpha i} = \int_{\delta^-}^{\delta^+} S_{\alpha i} N^{\pm}(\alpha_3) d\alpha_3, \quad T_{33} = \int_{\delta^-}^{\delta^+} S_{33} d\alpha_3, \quad \hat{T}^{\pm}_{\nu\kappa} = \int_{\delta^-}^{\delta^+} q_{\kappa} N^{\pm}(\alpha_3) d\alpha_3, \quad \kappa = \nu, t, 3 \end{gathered} \tag{2.4}$$

where S_{ij} are the components of the Piola–Kirchhoff symmetric stress tensor and b_{ijlm} are the stiffness characteristics of the material. Note that $E^{-}_{33} = E^{+}_{33} = E_{33}$ and $b_{\alpha\beta\gamma 3} = b_{\alpha 333} = 0$ are taken in formulae (2.3).

Taking account of the fact that the displacements, the deformations and the resulting stresses are independent functional variables, we represent a variation of the Hu–Washizu functional in the form

$$\begin{gathered} \delta J_{HW} = -\iint_S \left[\sum_{\pm} \left(\sum_{i+j<6} \left\{ \left[T^{\pm}_{ij} - \sum_{l+m<6} (D^{0+}_{ijlm} E^{-}_{lm} + D^{1+}_{ijlm} E^{+}_{lm}) - \underline{D^{+}_{ij33} E_{33}} \right] \delta E^{\pm}_{ij} + \right. \right. \right. \\ \left. \left. + (E^{\pm}_{ij} - e^{\pm}_{ij} - \eta^{\pm}_{ij}) \delta T^{\pm}_{ij} - T^{\pm}_{ij} (\delta e^{\pm}_{ij} + \delta \eta^{\pm}_{ij}) \right\} + \sum_i (\pm p^{\pm}_i) \delta v^{\pm}_i \right) + \\ + \left[T_{33} - \sum_{l+m<6} (D^{-}_{33lm} E^{-}_{lm} + D^{+}_{33lm} E^{+}_{lm}) - D_{3333} E_{33} \right] \delta E_{33} + \\ \left. + (E_{33} - e_{33} - \eta_{33}) \delta T_{33} - T_{33} (\delta e_{33} + \delta \eta_{33}) \right] A_1 A_2 \bar{\zeta}_1 \bar{\zeta}_2 d\alpha_1 d\alpha_2 - \\ - \oint_{\Gamma} \sum_{\pm} (\hat{T}^{\pm}_{\nu\nu} \delta v^{\pm}_{\nu} + \hat{T}^{\pm}_{\nu t} \delta v^{\pm}_t + \hat{T}^{\pm}_{\nu 3} \delta v^{\pm}_3)(1 + k_N \bar{\delta}) ds \end{gathered} \tag{2.5}$$

where

$$D^{-}_{ij33} = D^{0-}_{ij33} + D^{1-}_{ij33}, \quad D^{+}_{ij33} = D^{0+}_{ij33} + D^{1+}_{ij}, \quad D_{3333} = D^{-}_{3333} + D^{+}_{3333}$$

The complete relations of the generalized Hooke's law

$$S_{ij} = \sum_{l,m} b_{ijlm}\varepsilon_{lm} \tag{2.6}$$

can be used to calculate the components of the Piola–Kirchhoff symmetric stress tensor. However, in the analysis of shells made of incompressible materials or shells which are very similar to them as regards the characteristics of the materials for which Poisson's ratios are close to 1/2 [5, 10] and, also, with the aim of overcoming so-called Poisson wedging [8, 16], we shall approximately put $b_{\alpha\beta33} = 0$ in relations (2.6). At the same time, the equation for the transverse normal stress is used in an unchanged form, that is, $b_{\alpha\beta33} \neq 0$. This means that the underlined term in formula (2.5) has to be omitted. As a result, we arrive at an unsymmetrical rigidity matrix [13] which, however, does not introduce any substantial corrections into the numerical realization of the contact problem using a mixed finite-element model.

3. THE MODIFIED HU–WASHIZU FUNCTIONAL FOR SOLVING A CONTACT PROBLEM OF A TIMOSHENKO-TYPE SHELL WITH A RIGID PUNCH

To be specific, we will now assume that the contact of a shell with an absolutely rigid planar punch occurs over a part of the external surface of the shell S_c^+ and, moreover, that there is no friction in the contact region. We will write the conditions for non-penetration of the contacting bodies and the non-positiveness of the contact pressure q_c^+ in the form

$$g^+ - \mathbf{v}^+ \cdot \mathbf{n} \geq 0, \quad q_c^+ \leq 0 \tag{3.1}$$

where $g^+(\alpha_1, \alpha_2)$ is the initial clearance, that is, the shortest distance from a certain point of the shell $M^+(\alpha_1, \alpha_2)$ belonging to the surface of assumed contact S_c^+ to the punch, and $\mathbf{n}$ is the unit vector of the normal to the plane of the punch.

It is necessary to supplement inequalities (3.1) with the condition that the contact pressure is determined at points which come into contact with the rigid punch, that is, the following equality must be satisfied

$$q_c^+(g^+ - \mathbf{v}^+ \cdot \mathbf{n}) = 0 \tag{3.2}$$

In order to solve the problem of the contact interaction of a shell with a rigid punch, we will consider a modified Hu–Washizu functional which augments the functional (2.2) with a term, which is responsible for satisfying contact conditions (3.1) and (3.2) and yet another term [1] which is associated with the regularization of the problem

$$J = J_{HW} + \iint_{S_c^+} \left[\lambda(g^+ - \mathbf{v}^+ \cdot \mathbf{n}) - \frac{1}{2\varepsilon}\lambda^2\right] dS \tag{3.3}$$

where λ is a Lagrange multiplier (the contact pressure) and ε is a regularizing parameter. Note that the existence of a regularizing term in functional (3.3) implies the replacement of an absolutely rigid punch by a set of continuously distributed springs with a stiffness ε. The limiting case when $\varepsilon \to \infty$ corresponds to the classical method of Lagrange multipliers.

A formula for the variation of the modified Hu–Washizu functional will be required later. Taking account of expression (2.5), this formula can be represented in the form

$$\delta J = \delta J_{HW} + \iint_{S_c^+} \left[\left(g^+ - \mathbf{v}^+ \cdot \mathbf{n} - \frac{1}{\varepsilon}\lambda\right)\delta\lambda - \lambda \mathbf{n}\delta\mathbf{v}^+\right] dS \tag{3.4}$$

4. AN ALGORITHM FOR THE NUMERICAL SOLUTION OF THE CONTACT PROBLEM USING A MIXED FINITE-ELEMENT METHOD

A variation of the modified Hu–Washizu functional (3.4) for an element of a shell in its dimensionless curvilinear coordinates ξ_1 and ξ_2 can be written in matrix form as follows.

$$\delta J^{el} = -\int_{-1}^{1}\int_{-1}^{1}\Big[(\mathbf{T}-\mathbf{D}\mathbf{E})^T\delta\mathbf{E} + (\mathbf{E}-\mathbf{e}-\boldsymbol{\eta})^T\delta\mathbf{T} - \mathbf{T}^T(\delta\mathbf{e}+\delta\boldsymbol{\eta}) +$$

$$+ (\mathbf{P}+\lambda\mathbf{m})^T\delta\mathbf{v} - \left(g^+ - \mathbf{m}^T\mathbf{v} - \frac{1}{\varepsilon}\lambda\right)d\lambda\Big]\mu^{el}d\xi_1 d\xi_2 - \oint_{\Gamma^{el}}\hat{\mathbf{T}}_\Gamma^T\delta\mathbf{v}_\Gamma(1+k_N\bar{\delta})ds$$

$$\begin{aligned}
\mathbf{v}^T &= [v_1^- v_1^+ v_2^- v_2^+ v_3^- v_3^+], \quad \mathbf{v}_\Gamma^T = [v_\nu^- v_\nu^+ v_t^- v_t^+ v_3^- v_3^+] \\
\mathbf{E}^T &= [E_{11}^- E_{11}^+ E_{22}^- E_{22}^+ 2E_{12}^- 2E_{12}^+ 2E_{13}^- 2E_{13}^+ 2E_{23}^- 2E_{23}^+ E_{33}] \\
\mathbf{e}^T &= [e_{11}^- e_{11}^+ e_{22}^- e_{22}^+ 2e_{12}^- 2e_{12}^+ 2e_{13}^- 2e_{13}^+ 2e_{23}^- 2e_{23}^+ e_{33}] \\
\boldsymbol{\eta}^T &= [\eta_{11}^- \eta_{11}^+ \eta_{22}^- \eta_{22}^+ 2\eta_{12}^- 2\eta_{12}^+ 2\eta_{13}^- 2\eta_{13}^+ 2\eta_{23}^- 2\eta_{23}^+ \eta_{33}] \\
\hat{\mathbf{T}}^T &= [T_{11}^- T_{11}^+ T_{22}^- T_{22}^+ T_{12}^- T_{12}^+ T_{13}^- T_{13}^+ T_{23}^- T_{23}^+ T_{33}] \\
\mathbf{T}^T &= [\hat{T}_{\nu\nu}^- \hat{T}_{\nu\nu}^+ \hat{T}_{\nu t}^- \hat{T}_{\nu t}^+ \hat{T}_{\nu 3}^- \hat{T}_{\nu 3}^+], \quad \mathbf{P}^T = [-p_1^- p_1^+ - p_2^- p_2^+ - p_3^- p_3^+] \\
\mathbf{m}^T &= [0 n_1 0 n_2 0 n_3], \quad \mu^{el} = A_1 A_2 \bar{\zeta}_1 \bar{\zeta}_2 d_1^{el} d_2^{el}
\end{aligned} \tag{4.1}$$

where $2d_1^{el}$ and $2d_2^{el}$ are the lengths of an element in the directions α_1 and α_2, $\mathbf{v}$ is a column matrix of the displacements, $\mathbf{v}_\Gamma$ is a column matrix of the displacements of the boundary contour of an element Γ^{el}, $\mathbf{E}$ is a column matrix of the independently introduced deformation of the faces of the shell, $\mathbf{e}$ and $\boldsymbol{\eta}$ are column matrices characterizing the deformation relations (1.6) and (1.7), $\mathbf{T}$ is a column matrix of the resulting stresses, $\hat{\mathbf{T}}_\Gamma$ is a column matrix of the resulting loads acting on the boundary of an element Γ^{el}, $\mathbf{P}$ is a column matrix of the surface loads, and $\mathbf{D}$ is an 11×11 unsymmetrical matrix of the coefficients of elasticity, the elements of which are determined from relations (2.3) taking into account the assumptions [5, 8, 10, 13] made for calculating incompressible materials and, also, with the aim of overcoming Poisson wedging.

In the functional (4.1), the column matrices $\mathbf{v}$, $\mathbf{E}$ and $\mathbf{T}$ and the Lagrange multiplier λ are independent functional variables and it is therefore necessary to use independent approximations for them in an element. For the displacements and the Lagrange multiplier, we shall use the standard bilinear approximation

$$\mathbf{v} = \sum_r N_r \mathbf{v}_r, \quad \lambda = \sum_r N_r \lambda_r, \quad r = 1, \ldots, 4 \tag{4.2}$$

where $\mathbf{v}_r^T = [v_{1r}^- v_{1r}^+ v_{2r}^- v_{2r}^+ v_{3r}^- v_{3r}^+]$ are column matrices of the mesh-point displacements, $N_r(\xi_1, \xi_2)$ are linear functions of the form and λ_r are the values of the Lagrange multiplier at the mesh points of the element.

For the deformations, according to the method of double approximation [2, 17], which has been extended to the case where transverse compression is taken into account [7, 13], we have yet more simple formulae

$$\mathbf{E} = \sum_{r_1, r_2} \mathbf{Q}^{r_1 r_2} \mathbf{E}^{r_1 r_2} \xi_1^{r_1} \xi_2^{r_2} \tag{4.3}$$

Here,

$$\begin{aligned}
(\mathbf{E}^{00})^T &= [E_{11}^{-00} E_{11}^{+00} E_{22}^{-00} E_{22}^{+00} 2E_{12}^{-00} 2E_{12}^{+00} 2E_{13}^{-00} 2E_{13}^{+00} 2E_{23}^{-00} 2E_{23}^{+00} E_{33}^{00}] \\
(\mathbf{E}^{01})^T &= [E_{11}^{-01} E_{11}^{+01} 2E_{13}^{-01} 2E_{13}^{+01} E_{33}^{01}], \quad (\mathbf{E}^{10})^T = [E_{22}^{-10} E_{22}^{+10} 2E_{23}^{-10} 2E_{23}^{+10} E_{33}^{10}]
\end{aligned} \tag{4.4}$$

$$
\mathbf{Q}^{01} = \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1 \end{bmatrix}, \quad \mathbf{Q}^{10} = \begin{bmatrix} 0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}, \quad \mathbf{Q}^{11} = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix}
$$

where $\mathbf{Q}^{00}$ is an 11×11 unit matrix, $\mathbf{E}^{00}$ is a column matrix which characterizes the homogeneous form of deformation of an element, $\mathbf{E}^{01}$ and $\mathbf{E}^{10}$ are column matrices, characterizing the inhomogeneous forms of deformation and $\mathbf{E}^{11} = [E_{33}^{11}]$ is a 1×1 matrix, the introduction of which simplifies the matrix calculations here and henceforth in this section, the superscripts r_1 and r_2 take the values 0 and 1. We also note that the different nature of the approximation of the components of the Green–Lagrange strain tensor ensures the required number of degrees of freedom which, in turn, is responsible for the correct representation of the displacements of an element as a rigid whole [7, 13].

For the resulting stresses, we adopt the approximation, which is analogous to (4.3) and (4.4),

$$
\begin{aligned}
&\mathbf{T} = \sum_{r_1, r_2} \mathbf{Q}^{r_1 r_2} \mathbf{T}^{r_1 r_2} \xi_1^{r_1} \xi_2^{r_2}, \quad \mathbf{T}^{11} = [T_{33}^{11}] \\
&(\mathbf{T}^{00})^T = [T_{11}^{-00} T_{11}^{+00} T_{22}^{-00} T_{22}^{+00} T_{12}^{-00} T_{12}^{+00} T_{13}^{-00} T_{13}^{+00} T_{23}^{-00} T_{23}^{+00} T_{33}^{00}] \\
&(\mathbf{T}^{01})^T = [T_{11}^{-01} T_{11}^{+01} T_{13}^{-01} T_{13}^{+01} T_{33}^{01}] \quad (\mathbf{T}^{10})^T = [T_{22}^{-10} T_{22}^{+10} T_{23}^{-10} T_{23}^{+10} T_{33}^{10}]
\end{aligned}
\tag{4.5}
$$

Introducing the approximations (4.2)–(4.5) into formula (4.1) and using the standard variational procedure of a mixed model of the finite-element method, we arrive at the following equations for the equilibrium of an element

$$
\begin{aligned}
&\mathbf{E}^{r_1 r_2} = (\mathbf{Q}^{r_1 r_2})^T (\mathbf{L}^{r_1 r_2} + \mathbf{A}^{r_1 r_2} \mathbf{U}) \mathbf{U}, \quad \mathbf{T}^{r_1 r_2} = (\mathbf{Q}^{r_1 r_2})^T \mathbf{D} \mathbf{Q}^{r_1 r_2} \mathbf{E}^{r_1 r_2} \\
&\sum_{r_1 r_2} \frac{1}{3^{r_1 + r_2}} [(\mathbf{L}^{r_1 r_2} + 2\mathbf{A}^{r_1 r_2} \mathbf{U})^T \mathbf{Q}^{r_1 r_2} \mathbf{T}^{r_1 r_2} + \mathbf{B}^{r_1 r_2} \mathbf{\Lambda}] = \mathbf{F}
\end{aligned}
\tag{4.6}
$$

where $\mathbf{U}^T = [\mathbf{v}_1^T \mathbf{v}_2^T \mathbf{v}_3^T \mathbf{v}_4^T]$ is the column matrix of the mesh-point displacements of an element, $\mathbf{\Lambda}^T = [\lambda_1 \lambda_2 \lambda_3 \lambda_4]$ is the column matrix of the mesh-point values of the Lagrange multiplier, $\mathbf{F}$ is the column matrix of the mesh-point loads, $\mathbf{B}^{r_1 r_2}$ are 24×4 matrices corresponding to the contact interaction of an element, $\mathbf{L}^{r_1 r_2}$ are 11×24 matrices characterizing the linear components of the strain tensor (1.6) and (1.7), $\mathbf{A}^{r_1 r_2}$ are $11 \times 24 \times 24$ three-dimensional matrices characterizing the non-linear components of the strain tensor (1.6) and (1.7) and, at the same time, $\mathbf{A}^{r_1 r_2}\mathbf{U}$ are 11×24 matrices, the elements of which are calculated using the formulae

$$
(\mathbf{A}^{r_1 r_2} \mathbf{U})_{pq} = \sum_s A_{pqs}^{r_1 r_2} U_s; \quad p = 1, \ldots, 11; \quad q, s = 1, \ldots, 24
$$

Equations (4.6) are supplemented by relations which, according to formulae (3.1)–(3.3), are responsible for the contact conditions being satisfied. The conditions

$$
g_\rho^+ - \mathbf{m}_\rho^T \mathbf{v}_\rho - \frac{1}{\varepsilon} \lambda_\rho = 0, \quad \lambda_\rho \le 0 \tag{4.7}
$$

must be satisfied in the contact zone ($\rho \in I_c$, where $I_c \subset \{1, 2, 3, 4\}$) and, outside the contact zone ($\rho \notin I_c$), the conditions

$$g_\rho^+ - \mathbf{m}_\rho^T \mathbf{v}_\rho \geq 0, \quad \lambda_\rho = 0 \tag{4.8}$$

must be satisfied, where g_r^+ are the values of the clearance at the mesh points of an element, and $\mathbf{m}_r^T = [0n_{1r} 0n_{2r} 0n_{3r}]$ are column matrices characterizing the unit vectors of the normal to the plane of the punch at the mesh points of an element ($r = 1, \ldots, 4$). We also introduce the column matrices of the clearances at the mesh points of an element $\mathbf{G}^T = [g_1^+ g_2^+ g_3^+ g_4^+]$.

We will use an incremental approach to solve Eqs (4.6), taking into account the constraints (4.7) and (4.8), representing the required functions, the mesh-point loads and clearances in the form

$$\begin{aligned}
&{}^{t+\Delta t}\mathbf{E}^{r_1r_2} = {}^t\mathbf{E}^{r_1r_2} + \Delta\mathbf{E}^{r_1r_2}, \quad {}^{t+\Delta t}\mathbf{T}^{r_1r_2} = {}^t\mathbf{T}^{r_1r_2} + \Delta\mathbf{T}^{r_1r_2} \\
&{}^{t+\Delta t}\mathbf{U} = {}^t\mathbf{U} + \Delta\mathbf{U}, \quad {}^{t+\Delta t}\mathbf{F} = {}^t\mathbf{F} + \Delta\mathbf{F} \\
&{}^{t+\Delta t}\mathbf{\Lambda} = {}^t\mathbf{\Lambda} + \Delta\mathbf{\Lambda}, \quad {}^{t+\Delta t}\mathbf{G} = {}^t\mathbf{G} + \Delta\mathbf{G}
\end{aligned} \tag{4.9}$$

Quantities with the superscripts t and $t + \Delta t$ characterize the actual and final stress-strain state of the shell respectively, and $\Delta\mathbf{E}^{r_1r_2}$, $\Delta\mathbf{T}^{r_1r_2}$, $\Delta\mathbf{U}$, $\Delta\mathbf{F}$, $\Delta\mathbf{\Lambda}$ and $\Delta\mathbf{G}$ are incremental variables.

Next, introducing formulae (4.9) into relations (4.6) – (4.8) and taking into account the fact that the shell is at equilibrium in the actual state, we obtain the incremental equilibrium equations of an element

$$\begin{aligned}
&\Delta\mathbf{E}^{r_1r_2} = (\mathbf{Q}^{r_1r_2})^T({}^t\mathbf{M}^{r_1r_2} + \mathbf{A}^{r_1r_2}\Delta\mathbf{U})\Delta\mathbf{U}, \\
&(\Delta\mathbf{T}^{r_1r_2}) = (\mathbf{Q}^{r_1r_2})^T\mathbf{D}\mathbf{Q}^{r_1r_2}\Delta\mathbf{E}^{r_1r_2}\sum_{r_1, r_2}\frac{1}{3^{r_1+r_2}}\times \\
&\times[2(\mathbf{A}^{r_1r_2}\Delta\mathbf{U})^T\mathbf{Q}^{r_1r_2\,t}\mathbf{T}^{r_1r_2} + ({}^t\mathbf{M}^{r_1r_2} + 2\mathbf{A}^{r_1r_2}\Delta\mathbf{U})^T\mathbf{Q}^{r_1r_2}\Delta\mathbf{T}^{r_1r_2} + \mathbf{B}^{r_1r_2}\Delta\mathbf{\Lambda}] = \Delta\mathbf{F} \\
&{}^t\mathbf{M}^{r_1r_2} = \mathbf{L}^{r_1r_2} + 2\mathbf{A}^{r_1r_2\,t}\mathbf{U}
\end{aligned} \tag{4.10}$$

and, also, the incremental contact conditions when $\rho \in {}^{t+\Delta t}I_c$

$$\Delta g_\rho^+ - \mathbf{m}_\rho^T\Delta\mathbf{v}_\rho - \frac{1}{\varepsilon}\Delta\lambda_\rho = \begin{cases} 0, & \text{if} \quad \rho \in {}^tI_c \\ -{}^tg_+^+ + \mathbf{m}_\rho^{Tt}\mathbf{v}_\rho, & \text{if} \quad \rho \notin {}^tI_c \end{cases} \tag{4.11}$$

$$\Delta\lambda_\rho \leq \begin{cases} -{}^t\lambda_\rho, & \text{if} \quad \rho \in {}^tI_c \\ 0, & \text{if} \quad \rho \notin {}^tI_c \end{cases} \tag{4.12}$$

and, when $\rho \notin {}^{t+\Delta t}I_c$

$$\Delta g_\rho^+ - \mathbf{m}_\rho^T\Delta\mathbf{v}_\rho \geq \begin{cases} -\frac{1}{\varepsilon}{}^t\lambda_\rho, & \text{if} \quad \rho \in {}^tI_c \\ -{}^tg_+^+ + \mathbf{m}_\rho^{Tt}\mathbf{v}_\rho, & \text{if} \quad \rho \notin {}^tI_c \end{cases} \tag{4.13}$$

$$\Delta\lambda_\rho = \begin{cases} -{}^t\lambda_\rho, & \text{if} \quad \rho \in {}^tI_c \\ 0, & \text{if} \quad \rho \notin {}^tI_c \end{cases} \tag{4.14}$$

where ${}^tI_c$ and ${}^{t+\Delta t}I_c$ are certain subsets of the set $\{1, 2, 3, 4\}$.

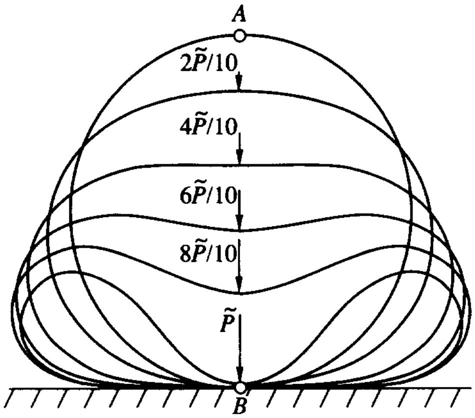

Fig. 2

Eliminating the incremental variables $\Delta\mathbf{E}^{r_1r_2}$ and $\Delta\mathbf{T}^{r_1r_2}$ from relations (4.10), we arrive at the following system of non-linear equations in the incremental variables $\Delta\mathbf{U}$ and $\Delta\mathbf{\Lambda}$

$$\sum_{r_1, r_2} \frac{1}{3^{r_1+r_2}}[2(\mathbf{A}^{r_1r_2}\Delta\mathbf{U})^T\mathbf{Q}^{r_1r_2\,t}\mathbf{T}^{r_1r_2} + \\ +({}^t\mathbf{M}^{r_1r_2} + 2\mathbf{A}^{r_1r_2}\Delta\mathbf{U})^T\mathbf{D}^{r_1r_2}({}^t\mathbf{M}^{r_1r_2} + \mathbf{A}^{r_1r_2}\Delta\mathbf{U})\Delta\mathbf{U} + \mathbf{B}^{r_1r_2}\Delta\mathbf{\Lambda}] = \Delta\mathbf{F} \tag{4.15}$$

which has to be solved together with the incremental contact conditions (4.11)–(4.14). The 11×11 matrices

$$\mathbf{D}^{r_1r_2} = \mathbf{Q}^{r_1r_2}(\mathbf{Q}^{r_1r_2})^T\mathbf{D}\mathbf{Q}^{r_1r_2}(\mathbf{Q}^{r_1r_2})^T$$

are introduced into formula (4.15) as a convenient notation.

The standard procedure for setting up the elements in the ensemble to obtain a system of non-linear equations in the global vector of the mesh-point variables is then used; this will not be discussed here.

The method of trial and error was used to solve the problem. The essence of this method is as follows. A starting approximation of the contact zone is initially specified, the non-linear system of equations (4.11), (4.14) and (4.15) is solved by the Newton–Raphson method and it is then checked that inequalities (4.12) and (4.13) are satisfied for each mesh point. If inequality (4.12) is not satisfied, the mesh point is moved out of the contact zone. In the case when inequality (4.13) is not satisfied, the mesh point is added to the contact zone.

5. RESULTS OF NUMERICAL CALCULATIONS

We will consider a circular arc which is pressed onto a rigid base by a concentrated force P, imposed on the upper section A, as shown in Fig. 2. The mechanical and geometrical characteristics of the arc were taken to be as follows: $E = 10^7$ N/mm^2, Poisson's ratio $\upsilon = 0$, the radius of the median circumference $R = 100$ mm, the thickness $h = 1$ mm and the width $b = 1$ mm. This problem has attracted attention (for example, see [18, 19]) from the point of view of studying the problem of the contact interaction of thin-walled structures which are subjected to large displacements and arbitrarily large rotations.

In view of the symmetry of the problem, one half of the arc, which was subdivided into 30 or 60 elements, was considered. Five loading steps were used into order to bring the low point of section A into "contact" with the high point of section B. In this case, the value $\tilde{P} = 88.40$, where $\tilde{P} = 120PR^2/(Eh^3b)$ was obtained for the compressing force. However, numerical experiments showed that a smaller number of increments can be used with an appropriate choice of the initial contact zone. The dependence of the dimensionless force on the dimensionless deflection of the median point of section A is represented by the solid curve in Fig. 3. The results of a calculation [19] using an incremental approach based on the well-known theory of axisymmetric shells of revolution of finite deflection [20], implemented numerically using the mixed finite-element method, are shown by the small open circles. Note that no data was given in [19] regarding the number of loading steps which was used.

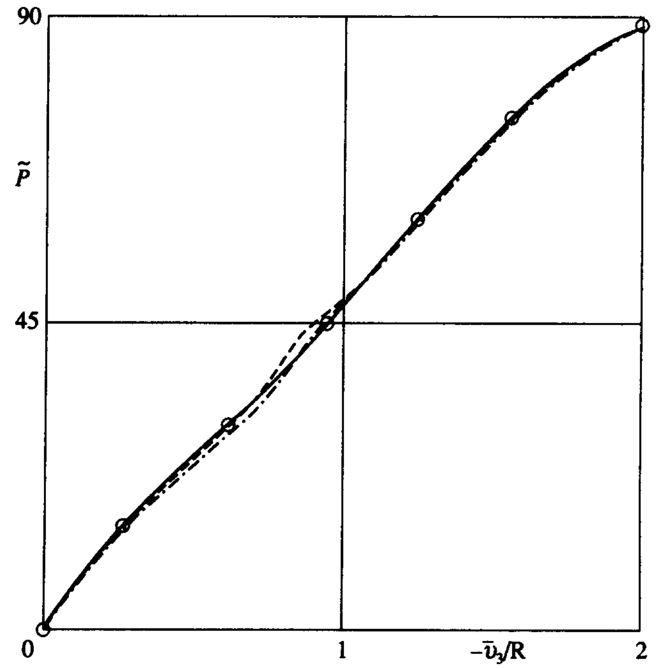

Fig. 3

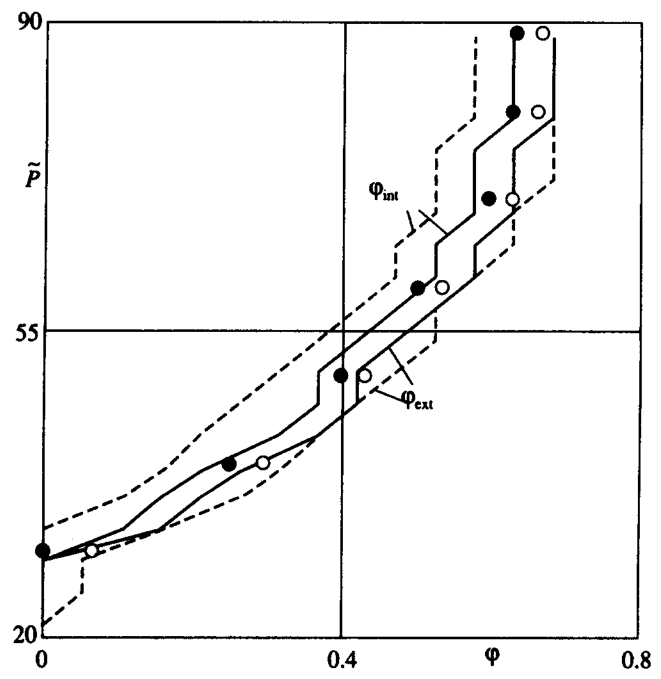

Fig. 4

Two values of the regularizing parameter: $\varepsilon = 10^3$ and $\varepsilon = 10^5$ were chosen for the calculations. It was found that the regularizing parameter had only a small effect on the form of the load–deflection curve. At the same time, a substantial dependence of the coordinates of the initial point of contact (φ_{int}) and the final point of contact (φ_{ext}) on the regularizing parameter ε was observed, since only a few mesh points come into contact with the base. The results of the numerical calculations are shown in Fig. 4 and Table 1, where the following notation is used: φ is the central angle, measured in an anticlockwise

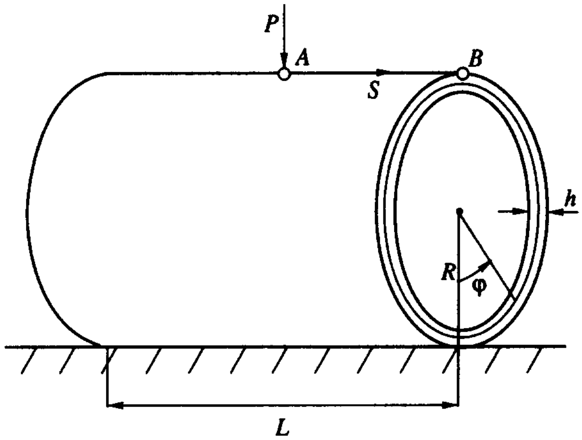

Fig. 5

Table 1

N^{el}	ε	$-\bar{v}_3^A/R$	$\tilde{Q}_c$	φ_{int}	φ_{ext}
30	10^3	1.986	88.86	0.628	0.733
60	10^3	1.992	88.51	0.576	0.681
30	10^5	1.983	88.86	0.628	0.733
60	10^5	1.991	88.51	0.628	0.681

Table 2

n	$0 \le s \le 30$	$s = 40$	$s = 50$	$s = 60$	$s = 70$	$s = 80$	$s = 90$
0	×	⊠	×	⊗	×	×	×
1	×	⊠	×	⊗	×	×	
2		$\boxed{\triangle}$		○			
3		$\boxed{\triangle}$	○	○			
4		□	$\textcircled{\triangle}$	○	○		
5	○	○	$\textcircled{\triangle}$	○	○		
6	○	○	$\textcircled{\triangle}$	○	○		
7	○	○	$\textcircled{\triangle}$	○	○	○	
8			△	△	○		
9	△	△	$\boxed{\triangle}$	△			
10	△	$\boxed{\triangle}$	$\boxed{\triangle}$	$\boxed{\triangle}$	△		
11	$\boxed{\triangle}$	$\boxed{\triangle}$	$\boxed{\triangle}$	$\boxed{\triangle}$	$\boxed{\triangle}$	△	
12	$\boxed{\triangle}$	$\boxed{\triangle}$	$\boxed{\triangle}$	$\boxed{\triangle}$	$\boxed{\triangle}$	$\boxed{\triangle}$	
13	□	□	□	□	□	□	
14	□	□	□	□	□	□	

direction from section B to section A, N^{el} is the number of elements and $\tilde{Q}_c$ is the dimensionless reaction of the contact pressure q_c^+.

$$\tilde{Q}_c = \frac{120R^2}{Eh^3 b}\iint_{S_c^+} q_c^+ dS$$

The solid curves in Fig. 4 correspond to $\varepsilon = 10^5$, the dashed curves correspond to $\varepsilon = 10^3$ and the results of the calculations by Noor and Kim [19] are represented by the open circles (φ_{ext}) and dark circles (φ_{int}). Note that the reaction of the contact pressure $\tilde{Q}_c$ is found with high accuracy, since a fairly similar value was indicated above in the case of a dimensionless compressive force $\tilde{P}$.

As a second example, we consider a circular cylindrical shell which is pressed onto a rigid base by a concentrated force P imposed on the central section A (Fig. 5). The mechanical and geometrical characteristics of the shell are: $E = 10^7$ N/mm^2, $\upsilon = 0.3$, $R = 100$ mm, L = 200 mm and $h = 1$ mm.

In view of the symmetry of the problem, only one quadrant is considered, which is subdivided into 10×60 elements. Five loading steps were used and the value $\varepsilon = 10^3$ was adopted. Graphs of the dimensionless force $\tilde{P} = 120(1-\nu^2)PR^2/(Eh^3L)$ against the dimensionless deflection of the median points of section A (the dot-dash curve) and section B (the dashed curve) are shown in Fig. 3. It can be seen that they practically coincide with the solid curve for the example considered above. The mesh points which have come into contact with the base for the different values of the load are shown in Table 2. Here, s is the meridional coordinate, $\varphi = n\pi/60$ is the central angle and the symbols $\times$, $\bigcirc$, $\triangle$, $\square$ correspond to the following values of the compressive force: $\tilde{P}/4, \tilde{P}/2, 3\tilde{P}/4, \tilde{P} = 87.36$.

REFERENCES

1. ZHONG, Z. H., *Finite Element Procedures for Contact-impact Problems*. Oxford University Press, Oxford, 1993.
2. BATHE, K. J., *Finite Element Procedures*. Prentice Hall, New Jersey; 1996.
3. AHMAD, S., IRONS, B. M. and ZIENKIEWICZ, O. C., Analysis of thick and thin shell structures by curved elements. *Int. J. Numer. Meth. Eng.* 1970, **2**, 419–451.
4. NOOR, A. K., Bibliography of books and monographs on finite element technology. *Appl. Mech. Rev.* 1991, **44**, 307–317.
5. KULIKOV, G. M. and PLOTNIKOVA, S. V., A comparative analysis of two algorithms for the numerical solution of problems of the statics of multilayer, anisotropic shells of revolution. 2. Taking account of transverse compression. *Mekhanika Kompozitnykh Materialov*, 1993, **35**, 4, 435–446.
6. KULIKOV, G. M., Refined global approximation theory of multilayered plates and shells. *J. Eng. Mech.*, 2001, **127**, 119–125.
7. KULIKOV, G. M. and PLOTNIKOVA, S. V., Simple and effective elements based upon Timoshenko–Mindlin shell theory. *Comput. Meth. Appl. Mech. Eng.*, 2002, **191**, 1173–1187.
8. KULIKOV, G. M. and PLOTNIKOVA, S. V., Investigation of locally loaded multilayer shells by the mixed finite-element method. 2. The geometrically non-linear formulation. *Mekhanika Kompozitnykh Materialov*, 2002, **38**, 6, 815–826.
9. KULIKOV, G. M. and PLOTNIKOVA, S. V., Numerical solution of the contact problem for multilayer composite plates, *Vestn. Tambov. Gas. Tekhn. Univ.*, 1998, **4**, 4, 526–539.
10. KULIKOV, G. M. and PLOTNIKOVA, S. V., The contact problem for a multilayer, anisotropic shell of revolution. In *Applied Problems of the Mechanics of Thin-Walled Structures*. Izd. MGU, 2000, pp. 205–223.
11. GRIGOLYUK, E. I. and TOKKACHEV, V. M., *Contact Problems in the Theory of Plates and Shells*. Mashinostroyeniye, Moscow, 1980.
12. NOVOZHILOV, V. V., *The Theory of Elasticity*. Sudpromgiz, Leningrad, 1958.
13. KULIKOV, G. M. and PLOTNIKOVA, S. V., Non-linear strain-displacement equations exactly representing large rigid body motions, Part I. Timoshenko–Mindlin shell theory. *Comput. Meth. Appl. Mech. Eng.*, 2003, **192**, 851–875.
14. ISHLINSKII, A. Yu., *Orientation, Gyroscopes and Inertial Navigation*. Nauka, Moscow, 1976.
15. WASHIZU, K., *Variational Methods in Elasticity and Plasticity*. Pergamon Press, Oxford, 1982.
16. BISCHOFF, M., RAMM, E., On the physical significance of higher order kinematic and static variables in a three-dimensional shell formulation. *Intern. J. Solids Structures*, 2000, **37**, 46–47, 6933–6960.
17. GOLOVANOV, A. I. and KORNISHIN, M. S., *Introduction to the Finite-element Method for the Statics of Thin Shells*. Kazan. Fiz. Tekh. Inst. Akad. Nauk SSSR, Kazan, 1989.
18. SIMO, J. C., WRIGGERS, P., SCHWEIZERHOFF, K. H. and TAYLOR, R. L., Finite deformation postbuckling analysis involving inelasticity and contact constraints. *Proc. Intern. Conf. Innovative Methods for Non-linear Problems*. Pineridge Press, Swansea; 1984, 365–387.
19. NOOR, A. K. and KIM, K. O., Mixed finite element formulation for frictionless contact problems. *Finite Elements Anal. Design*, 1989, **4**, 315–332.
20. REISSNER, E., On the equations for finite symmetrical deflections of thin shells of revolution. *Progress in Applied Mechanics*. Macmillan, New York, 1963, 171–178.

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